Chapter 6

6.1
6.2 } Answers given in the book
6.3

6.4
$$F(s) = \frac{1}{(s-3)(s-2)(s+1)}$$
$$= \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+1} = \frac{1/4}{s-3} + \frac{-1/3}{s-2} + \frac{1/12}{s+1}$$

(1) Suppose convergence region is $Re(s) < -1$. This implies all the poles of $F(s)$ lie to the right of the convergence region. Thus $f(t)$ is non zero for $t<0$ only. Using the transform pair (see Example 6.5)
$$\frac{1}{s-a} \longleftrightarrow -e^{at}u(-t), \quad Re(s) < a$$
we obtain
$$f(t) = \left\{-\tfrac{1}{4}e^{3t} + \tfrac{1}{3}e^{2t} - \tfrac{1}{12}e^{-t}\right\}u(-t)$$

(2) Suppose the convergence region is $-1 < Re(s) < 2$. The pole at -1 lies to the left of the convergence region and thus gives rise to a non zero time function for $t>0$. The other two poles at 2 & 3 lie to the right of the convergence region (as before). Using the transform pairs
$$\frac{1}{s-a} \longleftrightarrow -e^{at}u(-t), \quad Re(s) < a$$
$$\frac{1}{s-a} \longleftrightarrow e^{at}u(t), \quad Re(s) > a$$
we obtain

$$f(t) = \left(-\tfrac{1}{4}e^{3t} + \tfrac{1}{3}e^{2t}\right)u(-t) + \tfrac{1}{12}e^{-t}u(t)$$

(3) Suppose the convergence region is $2 < Re(s) < 3$. Now the poles at -1 and 2 give rise to nonzero $f(t)$ for $t>0$ and the pole at 3 gives rise to nonzero $f(t)$ for $t<0$. Thus
$$f(t) = -\tfrac{1}{4}e^{3t}u(-t) + \left(-\tfrac{1}{3}e^{2t} + \tfrac{1}{12}e^{-t}\right)u(t)$$

(4) Finally suppose the convergence region is $Re(s) > 3$. In this case all the poles lie to the left of the convergence region which means $f(t)$ is nonzero for $t>0$ only. Thus
$$f(t) = \left(\tfrac{1}{4}e^{3t} - \tfrac{1}{3}e^{2t} + \tfrac{1}{12}e^{-t}\right)u(t).$$

6.5 In general, $Y(s) = H(s)X(s)$

where $H(s) = \dfrac{1/RC}{s + 1/RC}$
$$X(s) = \int_{-\infty}^{\infty} e^{-|t|}e^{-st}dt = \int_{-\infty}^{0} e^{t(1-s)}dt + \int_{0}^{\infty} e^{t(-1-s)}dt$$
$$= \frac{1}{1-s} + \frac{1}{1+s}, \quad -1 < Re(s) < 1$$
$$= \frac{2}{1-s^2}, \quad -1 < Re(s) < 1$$
$$\therefore Y(s) = \frac{2}{1-s^2}\cdot\frac{w_0}{s+w_0} = \frac{-2w_0}{(s+w_0)(s-1)(s+1)}, \quad w_0 = 1/RC$$

The convergence region for $Y(s)$ is the intersection of the convergence regions for $H(s)$ and $X(s)$, i.e., the common region of convergence for $H(s)$ and $X(s)$. $X(s)$ converges for $-1 < Re(s) < 1$ and $H(s)$ for $Re(s) > -w_0$. Thus $Y(s)$ converges for $-w_0 < Re(s) < 1$ if $-1 < -w_0$ or $Y(s)$ converges for $-1 < Re(s) < 1$ if $-w_0 < -1$. In either case the poles at $-w_0$ and -1 are to the left of the convergence region and the pole at 1 is to the right. Thus

$$Y(s) = \frac{-2w_0}{(s+w_0)(s-1)(s+1)} = \frac{A}{s+w_0} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$= \frac{2w_0}{1-w_0^2}\left(\frac{1}{s+w_0}\right) - \frac{w_0}{1+w_0}\left(\frac{1}{s-1}\right) + \frac{w_0}{w_0-1}\left(\frac{1}{s+1}\right)$$

Thus $y(t)$ is given by

$$y(t) = \left\{ +\frac{2w_0}{1-w_0^2} e^{-w_0 t} + \frac{w_0}{w_0-1} e^{-t} \right\} u(t) + \left(\frac{+w_0}{1+w_0}\right) e^{t} u(-t)$$

6.6

Assume an input $\cos w_0 t$ and an output of the form c, a constant. The transform of the input is thus

$$X(s) = \mathcal{L}\{\cos w_0 t\, u(t)\} = \frac{s}{s^2+w_0^2}. \text{ The}$$

transform of the output is

$$Y(s) = \mathcal{L}\{c\, u(t)\}.$$

$$Y(s) = c/s.$$

Thus the system has transfer function given by

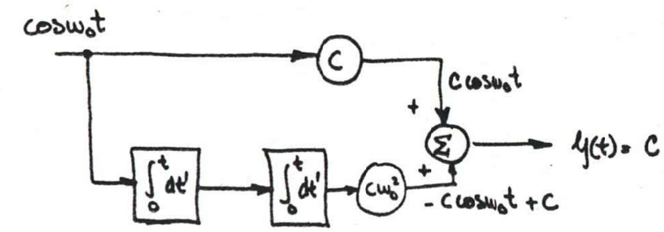
$$H(s) = \frac{Y(s)}{X(s)} = \frac{c/s}{\frac{s}{s^2+w_0^2}} = \frac{c(s^2+w_0^2)}{s^2}$$

$$= c + \frac{cw_0^2}{s^2}.$$

The impulse response function is thus

$$h(t) = c\,\delta(t) + cw_0^2\, t\, u(t).$$

A sketch of this system is shown below:



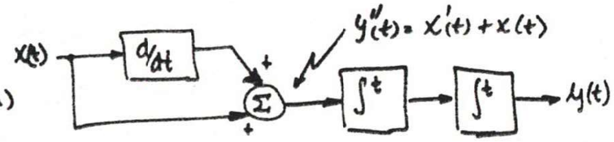
An input $\cos w_0 t$ produces $c\cos w_0 t$ in the upper branch. The lower branch produces $\frac{\sin w_0 t}{w_0}$ between the two integrators and $\left(-\frac{\cos w_0 t}{w_0^2} + \frac{1}{w_0^2}\right)$ at the output of the second integrator.

6.7 $\quad x(t) = e^{-t} u(t), \quad y(t) = t\, u(t)$

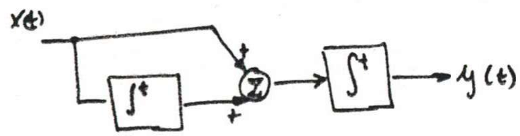
$$H(s) = Y(s)/X(s) = \frac{(1/s^2)}{1/(s+1)} = \frac{s+1}{s^2} = \frac{1}{s} + \frac{1}{s^2}$$

Any of the following circuits will work :

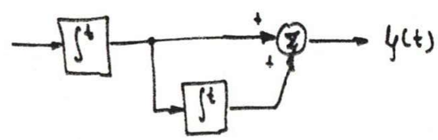
(a)



(b)



(C)



6.8

<u>system A</u> : $r(t) = e^{-at}u(t)$

$$y_A(t) = \int_0^t r(\tau)d\tau = \frac{1}{a}(1-e^{-at})u(t)$$

$$y_A(nT) = \frac{1}{a}(1-e^{-anT})u_n$$

$$= 5(1-e^{-.2n})u_n \qquad a=.2,\ T=1$$

<u>system B</u> : $r(nT) = e^{-anT}u_n$

$$y_B(nT) = \left[\sum_{m=0}^{n} r(mT)\right]u_n$$

$$= \left[\sum_{m=0}^{n}(e^{-aT})^m\right]u_n = \frac{1-e^{-aT(n+1)}}{1-e^{-aT}}$$

$$= \frac{1}{1-e^{-aT}}\left[1-e^{-aT}e^{-anT}\right]$$

$$\cong 5.52\,(1-.82e^{-.2n})u_n$$

<u>Comparison</u> :

n	0	1	2	3	4	5
$y_A(nT) \cong$	0	.90	1.65	2.25	2.75	3.15
$y_B(nT) \cong$	.99	1.82	2.44	3.04	3.48	3.86

A better discrete-time integrator using one delay. Suppose we implement Simpson's rule :

$$\int_{t_0}^{t_0+nT} x(\tau)d\tau \cong x(t_0)\cdot\frac{T}{2} + x(t_0+T)\cdot T + \cdots + x(t_0+(n-1)T)\cdot T + x(t_0+nT)\cdot\frac{T}{2}$$

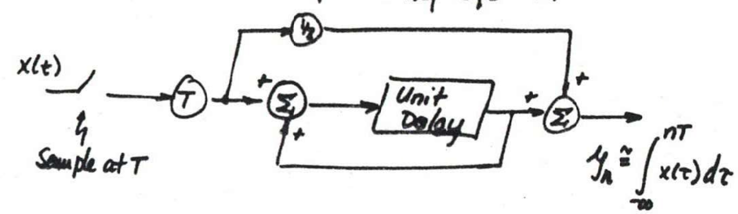
Let $t_0 \to -\infty$

$$\int_{-\infty}^{nT} x(\tau)d\tau \cong \left\{\cdots + x(0) + x(T) + \cdots + x((n-1)T) + \frac{x(nT)}{2}\right\}\cdot T$$

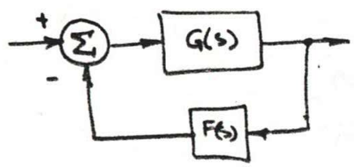
Let $w_n = \{\cdots x(0) + x(T) + \cdots + x((n-1)T)\}\cdot T$

then $\int_{-\infty}^{nT} x(\tau)d\tau \cong w_n + \frac{x(nT)}{2}\cdot T$

This can be realized by following system :



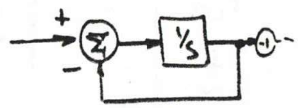
6.9



The transfer function of this system is $H(s) = \dfrac{G(s)}{1+G(s)H(s)}$

Now $G(s) = \dfrac{1}{s}$ (an integrator)

And $F(s) = \dfrac{-1/s}{1+1/s} = \dfrac{-1}{1+s}$



$$H(s) = \frac{1/s}{1-\frac{1}{s}\cdot\frac{1}{s+1}} = \frac{s+1}{s(s+1)-1}$$

$$= \frac{s+1}{s^2+s-1} \quad ; \text{ Poles at } \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$\therefore$ system unstable because of pole at $\frac{\sqrt{5}-1}{2}$ in right hand half plane.

Check by state variables:

$x_1(t) \triangleq$ output of left integrator

$x_2(t) \triangleq$ output of right integrator

$$\underline{A} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} ; \quad |A-\lambda I| = (-\lambda-1)(-\lambda)-1 = \lambda^2+\lambda-1$$

$$\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$ Again system is unstable.

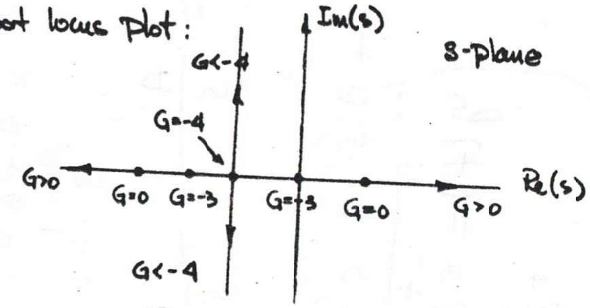
6.10 With $G=0$, there is a pole at $s=1$ & so system is unstable.

$$H(s) = \frac{G\,G(s)}{1-G\,G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1-\frac{G}{(s-1)(s+3)}} = \frac{1}{s^2+2s-3-G}$$

Poles at $\dfrac{-2 \pm \sqrt{4-4(-3-G)}}{2} = -1 \pm \sqrt{4+G}$

$G=0$: $-1\pm2 = 1;-3$	$G=-3$: $-1\pm1 = 0;-2$
$G=5$: $-1\pm3 = 2;-4$	$G=-4$: $-1\pm0 = -1;-1$
$G=12$: $-1\pm4 = 3;-5$	$G=-5$: $-1\pm j$
⋮	⋮

Root locus plot:

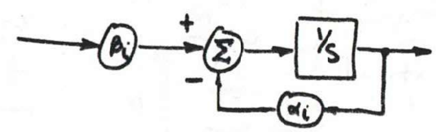


The system is stable for all $G < -3$.

6.11

Parallel form:

one section:



$$H_i(s) = \beta_i \frac{1/s}{1+\alpha_i \frac{1}{s}} = \frac{\beta_i}{s+\alpha_i}$$

$$H(s) = \sum_{i=1}^{n} H_i(s) = \frac{\beta_1}{s+\alpha_1} + \frac{\beta_2}{s+\alpha_2} + \cdots + \frac{\beta_n}{s+\alpha_n}$$

choose α_i to equal $-(i^{\text{th}}$ pole$)$; β_i to be the i^{th} coefficient in the partial fraction expansion of $H(s)$.

Cascade form :

$$Y(s) = X(s) \Big[\gamma_1 H_1(s) H_2(s) \cdots H_n(s) + \gamma_2 H_2(s) H_3(s) \cdots H_n(s) + \cdots + \gamma_{n-1} H_{n-1}(s) H_n(s) + \gamma_n H_n(s) \Big]$$

Thus

$$H(s) = \frac{\gamma_1}{(s+\alpha_1)(s+\alpha_2)\cdots(s+\alpha_n)} + \frac{\gamma_2}{(s+\alpha_2)\cdots(s+\alpha_n)} + \cdots + \frac{\gamma_{n-1}}{(s+\alpha_{n-1})(s+\alpha_n)} + \frac{\gamma_n}{s+\alpha_n}$$

$$= \frac{\gamma_1 + \gamma_2(s+\alpha_1) + \cdots + \gamma_n(s+\alpha_1)\cdots(s+\alpha_{n-1})}{(s+\alpha_1)(s+\alpha_2)\cdots(s+\alpha_n)}$$

Now choose $\gamma_1, \gamma_2, \ldots, \gamma_n$ so that the numerator equals $\frac{1}{b_n}(a_0 + a_1 s + \cdots + a_{n-1} s^{n-1})$. Choose α_i

as before to equal $-(i^{\text{th}}$ pole$)$. Thus

$$\gamma_n = \frac{a_{n-1}}{b_n} \quad (\text{may be zero})$$

$$\vdots$$

$$\gamma_1 = a_0 - \gamma_2\alpha_1 - \gamma_3\alpha_1\alpha_2 - \cdots - \gamma_n\alpha_1\alpha_2\cdots\alpha_{n-1}$$

(involves an iterative solution)

6.12 Answer given in book.