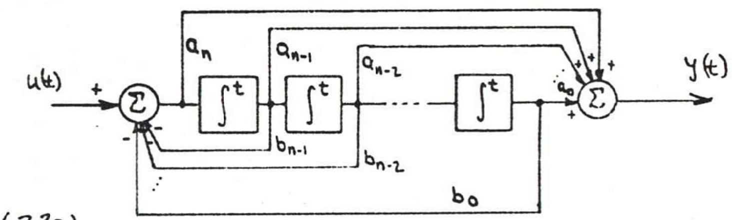
(3.29 cont)




(3.30)

$$e^A = \sum_{m=0}^{\infty} \frac{1}{m!} A^m \qquad e^B = \sum_{m=0}^{\infty} \frac{1}{m!} B^m$$

(a) $$e^A \cdot e^B = \left(I + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \dots\right) \cdot \left(I + B + \tfrac{1}{2}B^2 + \tfrac{1}{6}B^3 + \dots\right)$$

$$= I + (A+B) + \left(\tfrac{1}{2}A^2 + AB + \tfrac{1}{2}B^2\right) + \left(\tfrac{1}{6}A^3 + \tfrac{1}{2}A^2B + \tfrac{1}{2}AB^2 + \tfrac{1}{6}B^3\right) + \dots$$

compare with

$$e^{(A+B)} = I + (A+B) + \tfrac{1}{2}(A+B)^2 + \tfrac{1}{6}(A+B)^3 + \dots$$
$$= I + (A+B) + \tfrac{1}{2}(A^2 + AB + BA + B^2) + \tfrac{1}{6}(A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3) + \dots$$

These two expressions are identical iff $AB = BA$

Thus $e^{A+B} = e^A e^B$ iff A and B commute.

(b) Let $C = (e^A)^{-1}$

Then by definition $Ce^A = I = e^0 = e^{A-A} = e^{-A}e^A$ by the results of (a).

Hence $C = e^{-A}$ : ie $(e^A)^{-1} = e^{-A}$

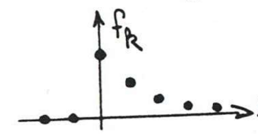


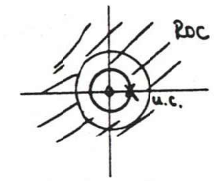
# Chapter 4

4.1 (a) $f_k = (1/2)^k, \; k \geq 0$

$$F(z) = \sum_0^{\infty} (1/2)^k z^{-k} = \sum \left(\tfrac{1}{2} z^{-1}\right)^k$$
$$= \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$\left|\frac{1}{2}z^{-1}\right| < 1 \Rightarrow \frac{1}{2} < |z|$

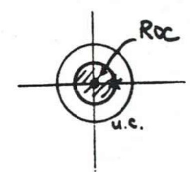



(b) $g_k = (1/2)^k, \quad k < 0$

$$G(z) = \sum_{-\infty}^{-1} \left(\tfrac{1}{2}\right)^k z^{-k} = \sum_1^{\infty} \left(\tfrac{1}{2}\right)^{-m} z^m$$
$$= \sum_1^{\infty} (2z)^m = \frac{2z}{1-2z} = \frac{-1}{1 - \frac{1}{2}z^{-1}}$$

$|2z| < 1 \Rightarrow |z| < 1/2$

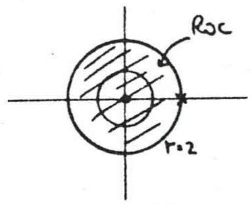



4.1 (continued)

(c) $h_k = 2^k, \quad k < 0$

$$H(z) = \sum_{-\infty}^{-1} 2^k z^{-k} = \sum_{1}^{\infty} \left(\tfrac{1}{2} z\right)^m = \frac{\frac{1}{2} z}{1 - \frac{1}{2} z}$$

$$\left|\tfrac{1}{2} z\right| < 1 \Rightarrow |z| < 2$$

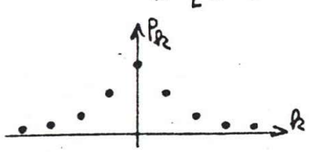

(d) $p_k = \left(\tfrac{1}{2}\right)^{|k|} \quad$ all $k$

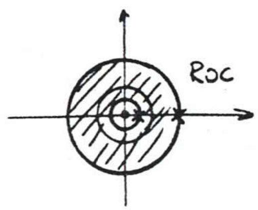
$$= \begin{cases} (1/2)^k & k \geq 0 \\ 2^k & k < 0 \end{cases}$$

$$= f_k + h_k$$

$$\therefore P(z) = F(z) + H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{-1}{1 - 2z^{-1}}$$

$$= \frac{-\frac{3}{2} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - 2z^{-1})}, \quad \tfrac{1}{2} < |z| < 2$$






4.1 (continued)

(e) $q_k = (1/2)^k + 2 \cdot (1/3)^k, \quad k \geq 0$

$$Q(z) = \dots = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$$

$\{z : |z| > 1/2\} \cap \{z : |z| > 1/3\}$ ie $|z| > 1/2$

(f) $r_k = 2^k + 3^k, \quad k \geq 0$

$$R(z) = \frac{1}{1 - 2z^{-1}} + \frac{1}{1 - 3z^{-1}} = \frac{2 - 5z^{-1}}{(1 - 2z^{-1})(1 - 3z^{-1})}$$

$|z| > 3$

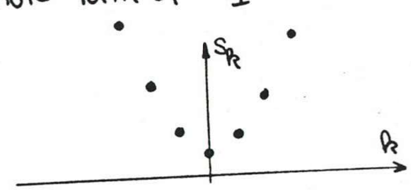
(g) $s_k = \begin{cases} 2^k & k \geq 0 \\ (1/3)^k & k < 0 \end{cases}$

$$S(z) = \frac{1}{1 - 2z^{-1}} - \frac{1}{1 - \frac{1}{3} z^{-1}}$$

$\{z : |z| > 2\} \cap \{z : |z| < 1/3\} = $ empty

$S(z)$ does not converge absolutely for any $z$.

note form of sequence:




This sequence does not have a valid transform.

4.1 (continued)

(h) $t_k = (1/2)^{|k|}$, all $k$

$$= \begin{cases} (1/2)^{k}, & k \geq 0 \\ (1/2)^{-k}, & k < 0 \end{cases}$$

formally, $T(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} - \dfrac{1}{1-\frac{1}{2}z^{-1}} \equiv 0$ !?

note ROC : $\{z : z > 1/2\} \cap \{z : z < 1/2\}$ = empty

ie $R_- = R_+$

This sequence does not have a valid transform

(i) $u_k = \cos \frac{k\pi}{8} = \frac{1}{2}\left[(e^{j\pi/8})^k + (e^{-j\pi/8})^k\right] \quad k \geq 0$

$$U(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\pi/8}z^{-1}} + \frac{1}{1-e^{-j\pi/8}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - e^{-j\pi/8}z^{-1} - e^{j\pi/8}z^{-1}}{1-(e^{j\pi/8}+e^{-j\pi/8})z^{-1}+z^{-2}}\right]$$

$$= \frac{1 - z^{-1}\cos \pi/8}{1 - 2\cos \pi/8 \, z^{-1} + z^{-2}} \qquad |z| > 1$$

(j) $v_k = 3\cos\left(\frac{k\pi}{8} + \frac{\pi}{8}\right) = \frac{3}{2}\left[e^{j\frac{\pi}{8}}(e^{j\frac{\pi}{8}})^k + e^{-j\frac{\pi}{8}}(e^{-j\frac{\pi}{8}})^k\right]$

$$V(z) = \frac{3}{2}\left[\frac{e^{j\pi/8}}{1-e^{j\pi/8}z^{-1}} + \frac{e^{-j\pi/8}}{1-e^{-j\pi/8}z^{-1}}\right]$$



4.1 (j) (continued)

$$V(z) = \frac{3}{2}\left[\frac{e^{j\pi/8} - z^{-1} + e^{-j\frac{\pi}{8}} - z^{-1}}{1-(e^{j\pi/8}+e^{-j\pi/8})z^{-1}+z^{-2}}\right]$$

$$= \frac{3\cos \pi/8 - 3z^{-1}}{1 - 2\cos\frac{\pi}{8}z^{-1} + z^{-2}} \qquad |z| > 1$$

alternate solution:

$$V(z) = 3\,Z\left\{\cos\left[(k+1)\pi/8\right]\right\} \qquad k \geq 0$$

$$= 3z\,Z\left\{\cos m\pi/8\right\} \qquad m = k+1;\ m \geq 1$$

$$= 3z\,Z\left\{\cos m\pi/8 - \delta_m\right\} \qquad m \geq 0$$

$$= 3z\left[\frac{1 - z^{-1}\cos \pi/8}{1 - 2\cos\frac{\pi}{8}z^{-1} + z^{-2}} - 1\right]$$

$$= \frac{3\cos \pi/8 - 3z^{-1}}{1 - 2\cos\frac{\pi}{8}z^{-1} + z^{-2}} \qquad |z| > 1$$

as above.

(k) $w_k = \begin{cases} 2^k, & k = 0, 2, 4, \ldots \\ 0, & \text{elsewhere} \end{cases}$

$$W(z) = \sum_{k=0,2,4,\ldots} 2^k z^{-k} = \sum_{m=0}^{\infty} (2z^{-1})^{2m}$$

$$= \sum_{m=0}^{\infty} (4z^{-2})^m = \frac{1}{1-4z^{-2}} \qquad |4z^{-2}| < 1$$

ie $2 < |z|$

4.2 (a) As given in the 1st printing:

$$y_k - 2y_{k-1} + y_{k+2} = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

write as

$$y_{k+2} + y_k - 2y_{k-1} = 1, \quad k \geq 0$$

ie

$$y_m + y_{m-2} - 2y_{m-3} = 1, \quad m \geq 2$$

$$Y(z)\left[1 + z^{-2} - 2z^{-3}\right] = \frac{z^{-2}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-2}}{(1+z^{-2}-2z^{-3})(1-z^{-1})}$$

As intended:

$$y_k - 2y_{k-1} + y_{k-2} = 1, \quad k \geq 0$$

$$Y(z)\left[1 - 2z^{-1} + z^{-2}\right] = \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{1}{(1-2z^{-1}+z^{-2})(1-z^{-1})}$$

(b) $$y_{k+2} - 2y_{k+1} + y_k = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Y(z)\left[z^2 - 2z + 1\right] = \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{1}{(z^2-2z+1)(1-z^{-1})} = \frac{z^{-2}}{(1-2z^{-1}+z^{-2})(1-z^{-1})}$$



4.2 (continued)

(c) $$y_k - 4y_{k-2} = \begin{cases} (1/2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Y(z)\left[1 - 4z^{-2}\right] = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) = \frac{1}{(1-4z^{-2})(1-\frac{1}{2}z^{-1})}$$

(d) $$y_{k+1} - y_{k-2} = \begin{cases} 2, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Y(z)\left[z - z^{-2}\right] = \frac{2}{1-z^{-1}}$$

$$Y(z) = \frac{2z^{-1}}{(1-z^{-3})(1-z^{-1})}$$

(e) $$y_{k+1} + 3y_k = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Y(z)\left[z + 3\right] = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$Y(z) = \frac{z^{-2}}{(1+3z^{-1})(1-z^{-1})^2}$$

(f) $$y_{k+1} - 5y_k = \begin{cases} \sin k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Y(z)\left[z - 5\right] = \frac{z^{-1}\sin(1)}{1 - 2z^{-1}\cos(1) + z^{-2}}$$

$$Y(z) = \frac{0.841\,z^{-1}}{1 - 1.081\,z^{-1} + z^{-2}}$$

4.3 (a) $y_k - a y_{k-1} = 0 \qquad y_0 = 1$

1st method: take $Z_u\{\cdot\}$ term-by-term on equation as it stands:

$$Y(z) - a\left[y_{-1} + z^{-1}Y(z)\right] = 0$$

$y_{-1}$ is found by setting $k=0$ to obtain

$$y_0 - a y_{-1} = 0 \Rightarrow y_{-1} = \frac{1}{a} y_0 = 1/a$$

now we have

$$Y(z) - a \cdot 1/a - a z^{-1} Y(z) = 0$$

$$Y(z)\left[1 - a z^{-1}\right] = 1$$

$$Y(z) = \frac{1}{1 - a z^{-1}}$$

for the initially relaxed system, we have

$$y_k - a y_{k-1} = x_k$$

ie $Y(z)\left[1 - a z^{-1}\right] = X(z)$

$$Y(z) = \frac{X(z)}{1 - a z^{-1}}$$

comparing these two expressions, we have

$$\frac{1}{1-a z^{-1}} \equiv \frac{X(z)}{1 - a z^{-1}}$$

ie $X(z) = 1$

$x_k = \delta_k$

Thus $y_k - a y_{k-1} = \delta_k, \qquad k \geq 0$



4.3 (a) (continued)

2nd method: replace $k$ by $k+1$:

$$y_{k+1} - a y_k = 0 \qquad y_0 = 1$$

$$z\left[Y(z) - y_0\right] - a Y(z) = 0$$

$$Y(z)\left[z - a\right] = z y_0 = z$$

$$Y(z) = \frac{z}{z-a} = \frac{1}{1-a z^{-1}} \equiv \frac{X(z)}{1 - a z^{-1}}$$

thus $X(z) = 1$

$x_k = \delta_k$ as before,

and from the original equation

$$y_k - a y_{k-1} = x_k$$

we have

$$y_k - a y_{k-1} = \delta_k \qquad k \geq 0$$

(b) $y_k - 2y_{k-1} + y_{k-2} = 0 \qquad y_{-1} = 0,\ y_0 = 1$

as in the second method above, we rewrite the difference equation in order to incorporate the given initial conditions in the transform:

$$y_{k+1} - 2y_k + y_{k-1} = 0$$

$$zY(z) - z y_0 - 2Y(z) + z^{-1}Y(z) + y_{-1} = 0$$

$$Y(z)\left(z - 2 + z^{-1}\right) = z y_0 - y_{-1} = z$$

ie $Y(z) = \dfrac{z}{z - 2 + z^{-1}} = \dfrac{1}{1 - 2z^{-1} + z^{-2}}$

4.3 (b) (continued)

comparing with

$$y_k - 2y_{k-1} + y_{k-2} = x_k$$

ie $Y(z)\left[1-2z^{-1}+z^{-2}\right] = X(z)$

$$Y(z) = \frac{X(z)}{1-2z^{-1}+z^{-2}}$$

we see that $X(z) = 1$ ; ie, $x_k = \delta_k$

hence

$$y_k - 2y_{k-1} + y_{k-2} = \delta_k \qquad k \geq 0$$

(c) following the same solution, we now have $y_{-1} = 1$, $y_0 = 0$ ; hence

$$Y(z) = \frac{zy_0 - y_{-1}}{z-2+z^{-1}} = \frac{-1}{z-2+z^{-1}}$$

$$= \frac{-z^{-1}}{1-2z^{-1}+z^{-2}}$$

from which $X(z) = -z^{-1}$

ie $x_k = -\delta_{k-1}$

(d) now with $y_{-1} = 1$, $y_0 = 1$,

$$Y(z) = \frac{z-1}{z-2+z^{-1}} = \frac{1-z^{-1}}{1-2z^{-1}+z^{-2}} \equiv \frac{X(z)}{1-2z^{-1}+z^{-2}}$$

$X(z) = 1-z^{-1}$: $\quad x_k = \delta_k - \delta_{k-1}$

Note how superposition may be applied to initial conditions



4.4 $f_{k+1} - 2f_k + f_{k-1} = \phi_k$

$$zF(z) - zf_0 - 2F(z) + z^{-1}F(z) + f_{-1} = \Phi(z)$$

$$F(z)\left[z-2+z^{-1}\right] = \Phi(z) + zf_0 - f_{-1}$$

$$F(z) = \frac{\Phi(z) + zf_0 - f_{-1}}{z-2+z^{-1}} = \frac{z^{-1}\Phi(z) - z^{-1}f_{-1} + f_0}{1-2z^{-1}+z^{-2}}$$

(a) $\phi_k = a^k$, $a \neq 1$, $k \geq 0$

$$\Phi(z) = \frac{1}{1-az^{-1}}$$

$$F(z) = \frac{z^{-1}}{(1-az^{-1})(1-z^{-1})^2} + \frac{f_0 - z^{-1}f_{-1}}{(1-z^{-1})^2}$$

the general solution is of the form

$$f_k = Aa^k + B + Ck \qquad k \geq 0$$

(b) $\phi_k = 1$, $k \geq 0$

$$\Phi(z) = \frac{1}{1-z^{-1}}$$

$$F(z) = \frac{z^{-1}}{(1-z^{-1})^3} + \frac{f_0 - z^{-1}f_{-1}}{(1-z^{-1})^2}$$

The general solution is of the form

$$f_k = A + Bk + Ck^2, \qquad k \geq 0$$

4.4 (continued)

(c) $\phi_k = k, \quad k \geq 0$

$$\Phi(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$F(z) = \frac{z^{-2}}{(1-z^{-1})^4} + \frac{f_0 - z^{-1} f_{-1}}{(1-z^{-1})^2}$$

The general solution is of the form

$$f_k = A + Bk + Ck^2 + Dk^3, \qquad k \geq 0$$

4.5 $\quad f_{k+2} = f_k + f_{k+1}, \quad k \geq 0$

$\mathcal{Z}_u\{\cdot\}$:

$$z^2 F(z) - f_0 z^2 - f_1 z = F(z) + zF(z) - zf_0$$

$$F(z)\left(z^2 - 1 - z\right) = z^2 f_0 + z(f_1 - f_0)$$

$$= z \quad \text{from } f_0 = 0,\ f_1 = 1$$

thus

$$F(z) = \frac{z}{z^2 - z - 1} = \frac{z}{\left(z - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(z - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)}$$

$$= \frac{\sqrt{\frac{1}{5}}\, z}{z - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)} - \frac{\sqrt{\frac{1}{5}}\, z}{z - \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)}$$

from which

$$f_k = \sqrt{\frac{1}{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^k - \sqrt{\frac{1}{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^k \quad k \geq 0$$

check: $f_0 = 0$ ✓ $\quad f_1 = 1$ ✓



4.5 (continued) (b)

(b)

$$\frac{f_k}{f_{k+1}} = \frac{\sqrt{\frac{1}{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]}{\sqrt{\frac{1}{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]}$$

$$= 2\,\frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}$$

$$\xrightarrow[k \to \infty]{} 2\,\frac{(1+\sqrt{5})^k}{(1+\sqrt{5})^{k+1}} = \frac{2}{1+\sqrt{5}}$$

4.6 (a) define the matrix $\underline{\underline{A}} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

then $\underline{A}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\underline{\underline{A}}^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$\vdots$

and in general it can be shown that

$$\underline{A}^k = \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix}$$

Replacing $k$ by $2k+1$, we have

$$\underline{\underline{A}}^{2k+1} = \begin{bmatrix} f_{2k} & f_{2k+1} \\ f_{2k+1} & f_{2k+2} \end{bmatrix}$$

4.6 (a) (continued)

Now with

$$A^{2k+1} = A^{k+1} \cdot A^{k}$$

$$= \begin{bmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{bmatrix} \cdot \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} f_k(f_{k-1}+f_{k+1}) & f_k^2 + f_{k+1}^2 \\ f_{k+1}f_{k-1}+f_k f_{k+2} & f_k f_{k+1} + f_{k+1} f_{k+2} \end{bmatrix} = \begin{bmatrix} f_{2k} & f_{2k+1} \\ f_{2k+1} & f_{2k+2} \end{bmatrix}$$

Equating the 1,2 component of the two matrices, we obtain

$$f_{2k+1} = f_k^2 + f_{k+1}^2 \qquad \text{(note error in 1st printing)}$$

(b) Let $f_k$ satisfy the difference equation

$$f_{k+2} = f_k + f_{k+1}$$

Then $f_{k+2}^2 - f_{k+1}^2 = (f_{k+2} - f_{k+1})(f_{k+2} + f_{k+1})$

$$= f_k \cdot f_{k+3}$$

(c) We wish to evaluate partial sums of the $\{f_k\}$. Let $g_k = \sum_{m=0}^{k} f_m$. Then the Z-transform of $g_k$



4.6 (c) (continued)

is given by

$$G(z) = \frac{1}{1-z^{-1}} F(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{(1-z^{-1}\alpha)(1-z^{-1}\beta)}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$

$$= \frac{-1}{1-z^{-1}} + \frac{\frac{\alpha}{(\alpha-1)(\alpha-\beta)}}{1-z^{-1}\alpha} + \frac{\frac{\beta}{(\beta-1)(\beta-\alpha)}}{1-z^{-1}\beta}$$

with the inverse transform

$$g_k = -1 + \frac{1}{\alpha-\beta}\left[\frac{\beta^{k+1}}{\alpha} - \frac{\alpha^{k+1}}{\beta}\right]$$

Note that $f_k = \frac{\alpha^k - \beta^k}{\alpha-\beta}$; thus $g_k = f_{k+2} - 1 = \sum_{m=0}^{k} f_m$

Now let $a_k$ be any sequence which satisfies

$$a_{k+2} = a_k + a_{k+1}.$$

Let $a_0 = u$ and $a_1 = v$

Then $a_k = u \cdot f_{k-1} + v \cdot f_k$

and $\sum_{m=0}^{k} a_m = u \sum_{m=0}^{k} f_{m-1} + v \sum_{m=0}^{k} f_m$

$$= u(f_{k+1} - 1 + f_{-1}) + v(f_{k+2} - 1)$$

$$= f_{k+1} \cdot u + (f_{k+2} - 1) \cdot v \qquad \text{with } f_{-1} = f_1 - f_0 = 1$$

In the parlor trick, let $(a_0 \ldots a_9)$ be the ten numbers with $a_0 = u$ and $a_1 = v$. The seventh number is

4.6 (c) (continued)

$$a_6 = f_5 \cdot u + f_6 \cdot v = 5u + 8v$$

and the sum is

$$\sum_{n=0}^{9} a_m = u \cdot f_{10} + v \cdot (f_{11} - 1)$$
$$= 55u + 11v = 11 a_6 .$$

4.7 (a) The first equation follows from the definition of specific heat:

$$c = \frac{1}{M} \frac{q}{\frac{d}{dt} T(t)}$$

and the second from the definition of thermal conductivity

$$k = \frac{qh}{A \Delta T}$$

(b) The general solution may be found by formulating the given equation in matrix form, as

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \vdots \\ \dot{T}_n \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} T_1(t) \\ T_2(t) \\ \vdots \\ T_n(t) \end{bmatrix}$$

with the solution

$$\underline{T}(t) = \underline{T}(0) \underline{\underline{e}}^{-\underline{\underline{A}}t}$$



4.7 (b) (continued)

For $n=2$, the roots of the characteristic equation for $\underline{\underline{A}}$ are 0 and -2. For $n=3$ the roots are $0, -1, -3$; for $n=4$, the roots are $0, -2, -2+\sqrt{2}, -2-\sqrt{2}$; and in general for higher order matrices one finds the eigenvalues of $\underline{\underline{A}}$ in order to solve the given matrix equation.

4.8 (a) $$\sum_{0}^{\infty} e^{-x(2n+1)} = e^{-x} \sum_{0}^{\infty} (e^{-2x})^n = \frac{e^{-x}}{1-e^{-2x}}$$

for $e^{-2x} < 1$; ie, $x > 0$

(b) $$\sum_{k=0}^{\infty} \alpha^k \sinh(kx) = \sum_{k=0}^{\infty} \alpha^k \left( \frac{e^{kx} - e^{-kx}}{2} \right)$$
$$= \frac{1}{2} \sum_{0}^{\infty} (\alpha e^{x})^k - \frac{1}{2} \sum_{0}^{\infty} (\alpha e^{-x})^k$$
$$= \frac{1}{2} \left[ \frac{1}{1-\alpha e^{x}} - \frac{1}{1-\alpha e^{-x}} \right] = \frac{\alpha \sinh x}{1 - 2\alpha \cosh x + \alpha^2}$$

or by using the properties

$$\sinh kx \xrightarrow{Z} \frac{z^{-1} \sinh x}{1 - 2z^{-1} \cosh x + z^{-2}}$$

$$\alpha^k \sinh kx \xrightarrow{Z} \frac{\alpha z^{-1} \sinh x}{1 - 2\alpha z^{-1} \cosh x + \alpha^2 z^{-2}}$$

evaluate at $z=1$ : $$\sum_{k=0}^{\infty} \alpha^k \sinh(kx) = \frac{\alpha \sinh x}{1 - 2\alpha \cosh x + \alpha^2}$$

4.9 Let $x(n)$ be the number of solutions in nonnegative integers $l_1$ and $l_2$ to the equation $n = l_1 + 2l_2$. The generating function for $x(n)$, following the discussion in the problem, is

(a) $$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-z^{-2}} = \frac{1}{(1-z^{-1})^2 (1+z^{-1})}$$

$$= \frac{1/2}{(1-z^{-1})^2} + \frac{-1/4}{1-z^{-1}} + \frac{1/4}{1+z^{-1}}$$

(c) Thus $x(n) = \frac{1}{2}(n+1) - \frac{1}{4} + \frac{1}{4}(-1)^n$

This sequence can satisfy infinitely many difference equations, but only one linear difference equation of order three (we know this from the form of $X(z)$). Thus we set

$$x(n+3) + a\,x(n+2) + b\,x(n+1) + c\,x(n) = y(n)$$

Taking Z-transforms through this equation and equating the result to $\frac{1}{(1+z^{-1})(1-z^{-1})^2}$, we obtain

$$a = -1, \quad b = -1, \quad c = 1, \quad \text{and } y(n) \equiv 0.$$

(b) Thus $x(n)$ satisfies the equation

$$x(n+3) - x(n+2) - x(n+1) + x(n) = 0$$



4.10 (a) $y_k - 2y_{k-1} + y_{k-2} = 1, \quad k \geq 0, \quad y_{-1} = y_{-2} = 1$

$$Y_u(z) - 2\left[z^{-1}Y_u(z) + y_{-1}\right] + z^{-2}Y_u(z) + z^{-1}y_{-1} + y_{-2} = \frac{1}{1-z^{-1}}$$

$$Y_u(z)\left(1 - 2z^{-1} + z^{-2}\right) = \frac{1}{1-z^{-1}} + (2 - z^{-1})y_{-1} - y_{-2}$$

$$= \frac{1}{1-z^{-1}} + 1 - z^{-1} = \frac{1 + (1-z^{-1})^2}{(1-z^{-1})}$$

$$Y_u(z) = \frac{1 + (1-z^{-1})^2}{(1-z^{-1})^3}$$

(b) $y_k - \frac{1}{4}y_{k-2} = (1/2)^k \quad k \geq 0, \quad y_{-1} = 0, \quad y_{-2} = 1$

$$Y_u(z) - \frac{1}{4}\left[z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}\right] = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y_u(z)\left(1 - \frac{1}{4}z^{-2}\right) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{4}$$

$$Y_u(z) = \frac{4 + 1 - \frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-2})(1-\frac{1}{2}z^{-1})} = \frac{5 - \frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2(1+\frac{1}{2}z^{-1})}$$

(c) $y_{k+2} - \frac{1}{4}y_k = \left(\frac{1}{2}\right)^k, \quad k \geq 0, \quad y_0 = y_1 = 1$

$$z^2 Y_u(z) - z^2 y_0 - z y_1 - \frac{1}{4}Y_u(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y_u(z)\left(z^2 - \frac{1}{4}\right) = \frac{1}{1-\frac{1}{2}z^{-1}} + z^2 y_0 + z y_1$$

$$Y_u(z) = \frac{1}{(z^2-\frac{1}{4})(1-\frac{1}{2}z^{-1})} + \frac{z^2+z}{z^2-\frac{1}{4}}$$

4.10 (continued)

(d) $y_{k+2} + y_{k+1} - y_k = 1 \qquad k \geq 0, \; y_0 = 1, \; y_1 = 2$

$$z^2 Y_u(z) - z^2 y_0 - z y_1 + z Y(z) - z y_0 - Y(z) = \frac{1}{1-z^{-1}}$$

$$Y_u(z)\left(z^2 + z - 1\right) = \frac{1}{1-z^{-1}} + (z^2+z)y_0 + z y_1$$

$$= \frac{1}{1-z^{-1}} + z^2 + 3z$$

$$Y_u(z) = \frac{1}{(z^2+z-1)(1-z^{-1})} + \frac{z^2+3z}{z^2+z-1}$$

(e) $y_{k+1} + y_k - y_{k-1} = 1 \qquad k \geq 0, \; y_0 = 1, \; y_{-1} = -1$

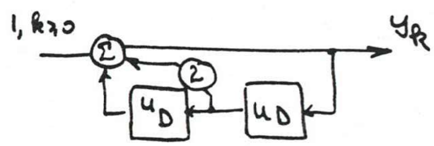
$$z Y_u(z) - z y_0 + Y_u(z) - \left[z^{-1} Y_u(z) + y_{-1}\right] = \frac{1}{1-z^{-1}}$$

$$Y_u(z)\left(z + 1 - z^{-1}\right) = \frac{1}{1-z^{-1}} + z y_0 + y_{-1}$$

$$= \frac{1}{1-z^{-1}} + z - 1 = \frac{1 + z(1-z^{-1})^2}{1-z^{-1}}$$

$$Y_u(z) = \frac{1 + z(1-z^{-1})^2}{(1-z^{-1})(z+1-z^{-1})}$$

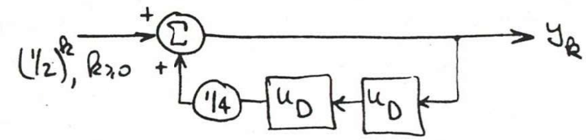
4.11 (a)



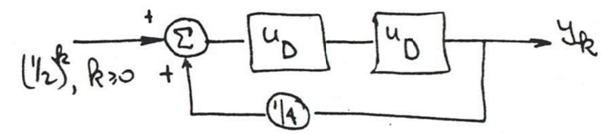



4.11 (continued)

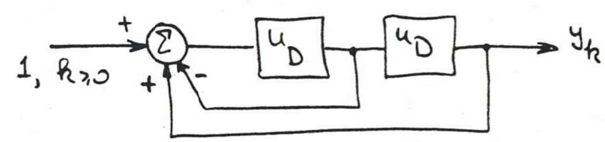
(b)



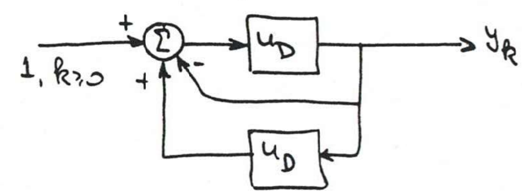

(c)




(d)




(e)



4.12 (a) PASCAL sample:

```
program difeq(input,output);

var y: array[-2..0] of real;
    k: integer;

begin
  y[-2] := 1;
  y[-1] := 1;
  writeln;
  writeln(' y(-2) =',y[-2]:7:2);
  writeln(' y(-1) =',y[-1]:7:2);
  for k:=0 to 10 do
    begin
      y[0] := 2*y[-1] - y[-2] +1;
      writeln(' y(',k:2,') =',y[0]:7:2);
      y[-2] := y[-1];
      y[-1] := y[0]
    end
end.
```

output:

```
y(-2) =    1.00
y(-1) =    1.00
y( 0) =    2.00
y( 1) =    4.00
y( 2) =    7.00
y( 3) =   11.00
y( 4) =   16.00
y( 5) =   22.00
y( 6) =   29.00
y( 7) =   37.00
y( 8) =   46.00
y( 9) =   56.00
y(10) =   67.00
```

(b) FORTRAN sample:

```
C       PROGRAM DIFEQ
C
        YKM2 = 1.0
        YKM1 = 1.0
        WRITE(6,610) YKM2, YKM1
  610   FORMAT(//' Y(-2) =',F7.2/' Y(-1) =',F7.2)
C
        DO 100  K=1,11
          YK = 2.0*YKM1 - YKM2 + 1.0
          I = K-1
          WRITE(6,620) I, YK
  620     FORMAT(' Y(',I2,') =',F7.2)
          YKM2 = YKM1
          YKM1 = YK
100     CONTINUE
C
        CALL EXIT
        END
```

output:

```
Y(-2) =    1.00
Y(-1) =    1.00
Y( 0) =    2.00
Y( 1) =    4.00
Y( 2) =    7.00
Y( 3) =   11.00
Y( 4) =   16.00
Y( 5) =   22.00
Y( 6) =   29.00
Y( 7) =   37.00
Y( 8) =   46.00
Y( 9) =   56.00
Y(10) =   67.00
```



4.13 (a) $A(z) = \left[\dfrac{1}{1-\frac{1}{2}z^{-1}}\right]^2 \qquad \frac{1}{2} < |z|$

(b) $B(z) = \left[\dfrac{1}{1-\frac{1}{2}z^{-1}}\right]^M \qquad \frac{1}{2} < |z|$

(c) $C(z) = 1 - \dfrac{1}{1-2z^{-1}} = \dfrac{-2z^{-1}}{1-2z^{-1}} \qquad 2 > |z|$

alt: $f(-k) \xrightarrow{z} \sum f(-k) z^{-k}$

$= \sum f(k) z^{k} = F(1/z)$

$\therefore \{2^{k}\}_{-\infty}^{0} \to z\{2^{-k}\}_{0}^{\infty}\Big|_{z \to z^{-1}} = \dfrac{1}{1-\frac{1}{2}z^{-1}}\Big|_{z\to z^{-1}} = \dfrac{1}{1-\frac{1}{2}z}$

$|z| > 1/2 \qquad |z| < 2$

as above.

(d) $\{d_k\} = \{(1/2)^k\}_0^{\infty} + \{(1/2)^k\}_{-\infty}^{0}$

$D(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} \cdot \dfrac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$

$\frac{1}{2} < |z| \cap \frac{1}{2} > |z|$ : does not converge for any values of $z$

(d') (better problem):

$\{(1/2)^k\}_0^{\infty} + \{(1/2)^{-k}\}_{-\infty}^{0} \xrightarrow{z} \dfrac{1}{1-\frac{1}{2}z^{-1}} \cdot \dfrac{-2z^{-1}}{1-2z^{-1}}$

$\frac{1}{2} < |z| < 2$

4.13 (continued)

(e) $e_k = \sum_{m=0}^{k} (1/2)^m, \quad k \geq 0$

$$E(z) = \frac{\frac{1}{1-1/2\, z^{-1}}}{1-z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

by direct evaluation:

$$e_k = \sum_{m=0} (1/2)^m = \frac{1-(1/2)^{k+1}}{1-1/2} = 2-(1/2)^k, \quad k \geq 0$$

$$E(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}$$

(f) $f_k = \cos[(k-2)\pi/8], \quad k \geq 2$ ; ie, $k-2 \geq 0$

$= \gamma_{k-2}$ where $\gamma_k = \cos k\pi/8, \quad k \geq 0$

$\therefore F(z) = \Gamma(z) \cdot z^{-2}$, where

$$\Gamma(z) = \mathcal{Z}\{\cos k\pi/8\} = \frac{1-z^{-1}\cos\pi/8}{1-2z^{-1}\cos\pi/8 + z^{-2}}$$

hence $F(z) = \dfrac{z^{-2} - z^{-3}\cos\pi/8}{1-2z^{-1}\cos\pi/8 + z^{-2}}$

(g) $g_k = k\cos k\theta, \quad k \geq 0$

$$G(z) = -z\frac{d}{dz}\frac{1-z^{-1}\cos\theta}{1-2z^{-1}\cos\theta + z^{-2}}$$



4.13 (g) (continued)

$$G(z) = -z\frac{d}{dz}\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{-z\left[(z^2-2z\cos\theta+1)(2z-\cos\theta) - (z^2-z\cos\theta)(2z-2\cos\theta)\right]}{(z^2-2z\cos\theta+1)^2}$$

$$\frac{-z\left[-z^2\cos\theta + 2z - \cos\theta\right]}{(z^2-2z\cos\theta+1)^2}$$

$$= \frac{z^{-1} - 2z^{-2} + z^{-3}\cos\theta}{(1-2z^{-1}\cos\theta + z^{-2})^2}$$

(h) $h_k = \{k a^k \cos k\theta\}_0^\infty$

$= a^k g_k$

Thus

$$H(z) = G(z)\Big|_{z^{-1} \to az^{-1}}$$

$$= \frac{az^{-1} - 2a^2z^{-2} + a^3z^{-3}\cos\theta}{(1-2az^{-1}\cos\theta + a^2z^{-2})^2}$$

4.14 partial fraction expansion for a-d :

$$\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{Az}{z-1/2} + \frac{Bz}{z-1/3}$$

$$A = \left.\frac{z}{z-1/3}\right|_{z=1/2} = \frac{1/2}{1/2-1/3} = \frac{1/2}{1/6} = 3$$

$$B = \left.\frac{z}{z-1/2}\right|_{z=1/3} = \frac{1/3}{1/3-1/2} = \frac{1/3}{-1/6} = -2$$

Thus

$$\frac{z^2}{(z-1/2)(z-1/3)} = \frac{3z}{z-1/2} - \frac{2z}{z-1/3}$$

$$= \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$$

(a) if ROC is $|z|<1/3$

$$a_k = -3(1/2)^k + 2(1/3)^k, \quad k<0$$

(b) if ROC is $\frac{1}{3}<|z|<1/2$

$$b_k = \begin{cases} 3\cdot(1/2)^k & k\geq 0 \\ 2\,(1/3)^k & k<0 \end{cases}$$



4.14 (cont)

(c) if ROC is $\frac{1}{2}<|z|$

$$c_k = 3\,(1/2)^k - 2\,(1/3)^k, \quad k\geq 0$$

(d) $D(z) = z^{-2}\,C(z)$

$$d_k = c_{k-2} = 3(1/2)^{k-2} - 2(1/3)^{k-2}, \quad k-2\geq 0$$

$$= 12\,(1/2)^k - 18\,(1/3)^k, \quad k\geq 2$$

(e) $E(z) = \dfrac{-z}{(z-1/2)(z-2)}$ ; $e_k = 0,\ k<0$

$$= \frac{2/3}{1-1/2\,z^{-1}} - \frac{2/3}{1-2z^{-1}}$$

$$e_k = \frac{2}{3}\left[\left(\tfrac{1}{2}\right)^k - 2^k\right] \quad k\geq 0$$

(f) $F(z) = \dfrac{-z}{(z-1/2)(z-2)}$ ; $f_k$ has finite energy

$$= \frac{2/3}{1-\frac{1}{2}z^{-1}} - \frac{2/3}{1-2z^{-1}}$$

$$f_k = \begin{cases} 2/3\cdot\left(\tfrac{1}{2}\right)^k & k\geq 0 \\ 2/3\cdot(2)^k & k<0 \end{cases}$$

$$= \frac{2}{3}\cdot\left(\frac{1}{2}\right)^{|k|}, \quad \text{all } k$$

4.14 (continued)

(g) $$G(z) = \frac{z^3}{(z-1/4)^2(z-1)} \qquad |z|<1/4$$

$$= \frac{(-1/3)z^2}{(z-1/4)^2} + \frac{(-4/9)z}{z-1/4} + \frac{(16/9)z}{z-1}$$

$$= \frac{-1/3}{(1-1/4z^{-1})^2} - \frac{4/9}{1-1/4z^{-1}} + \frac{16/9}{1-z^{-1}}$$

$$g_k = \frac{1}{3}(k+1)\left(\frac{1}{4}\right)^k + \frac{4}{9}\left(\frac{1}{4}\right)^k - \frac{16}{9}, \quad k<0$$

$$= \left(\frac{k}{3}+\frac{7}{9}\right)\left(\frac{1}{4}\right)^k - \frac{16}{9} \quad k<0$$

(h) PFE as above, $\frac{1}{4}<|z|<1$

$$h_k = \begin{cases} -\frac{1}{3}(k+1)\left(\frac{1}{4}\right)^k - \frac{4}{9}\left(\frac{1}{4}\right)^k & k\geq 0 \\ -\frac{16}{9} & k<0 \end{cases}$$

$$= \begin{cases} -\left(\frac{k}{3}+\frac{7}{9}\right)\left(\frac{1}{4}\right)^k & k\geq 0 \\ -\frac{16}{9} & k<0 \end{cases}$$

(i) PFE as above, $1<|z|$

$$i_k = -\left(\frac{k}{3}+\frac{7}{9}\right)\left(\frac{1}{4}\right)^k + \frac{16}{9}, \quad k\geq 0$$



Correct solution for part h:

$$G(z) = \frac{-7/9\, z}{z-\frac{1}{4}} + \frac{-1/12\, z}{(z-\frac{1}{4})^2} + \frac{16/9\, z}{z-1}$$

$$g_K = \frac{-7}{9}\left(\frac{1}{4}\right)^K - \frac{1}{12}K\left(\frac{1}{4}\right)^{K-1} \quad \text{for } K\geq 0$$

$$g_K = \frac{-16}{9} \text{ for } K<0$$

4.14 (continued)

(j) $$Q(z) = \frac{1}{(z-\frac{1}{2})^2(z-2)} \quad ; \{q_k\} \text{ has finite energy}$$

$$= \frac{-2/3}{(z-1/2)^2} - \frac{4/9}{z-1/2} + \frac{4/9}{z-2}$$

$$q_k = \begin{cases} -4/9\,(2)^{k-1} & k\leq 0 \\ -\frac{2}{3}(k-1)\left(\frac{1}{2}\right)^{k-2} - \frac{4}{9}\left(\frac{1}{2}\right)^{k-1}, & k>0 \end{cases}$$

ie $$q_k = \begin{cases} -\frac{2}{9}\cdot 2^k & k\leq 0 \\ \left(-\frac{8}{3}k+\frac{16}{9}\right)\left(\frac{1}{2}\right)^k & k>0 \end{cases}$$

4.15 (a) $v_n \xrightarrow{Z} V(z) = X(z)\cdot G(z)$

$v_{-n} \xrightarrow{Z} V(1/z) = X(1/z)\,G(1/z)$

$w_n \xrightarrow{Z} W(z) = V(1/z)\,G(z)$

$= X(1/z)\,G(1/z)\,G(z)$

$y_n = w_{-n} \longrightarrow Y(z) = W(1/z)$

$= X(z)\cdot G(z)\cdot G(1/z)$

$\triangleq X(z)\,H(z)$

thus $H(z) = G(z)\,G(1/z)$

4.15 (continued)

(b) yes, since $G(z)$ converges absolutely for $|z| > r_0$, where $r_0 < 1$; thus $G(1/z)$ converges absolutely for $|1/z| > r_0$, ie $|z| < r_0$

$\Rightarrow$ $H(z)$ converges absolutely for $r_0 < |z| < 1/r_0$, including the unit circle

(c) no: $H(z)$ has poles inside and outside the unit circle, and is stable; hence, it cannot be causal.

also, $g_k * g_{-k}$ is nonzero for $k < 0$.

4.16 $G(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)} = \frac{-2z}{(z-\frac{1}{2})(z-2)}$$

$$= \frac{(4/3)z}{z - 1/2} - \frac{(4/3)z}{z-2}$$

$g_k = (1/2)^k,\ k \geq 0$

$$h_k = \left\{\begin{array}{ll} 4/3 \cdot 2^k & k<0 \\ 4/3 (1/2)^k & k \geq 0 \end{array}\right\} = \frac{4}{3}(1/2)^{|k|},\ \text{all } k$$



4.17 $r_{xx}(k) = \sum_{m=-\infty}^{\infty} x_m x_{m-k}$

(a) ✓

(b) $r_{xx}(k) = \sum x_m y_{k-m}$ where $y_k = x_{-k}$

$= x_k * y_k$

$= x_k * x_{-k}$

(c) $R_{xx}(z) = X(z) \cdot X(1/z)$

(d) if $x_k = (1/2)^k,\ k \geq 0$

$X(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$

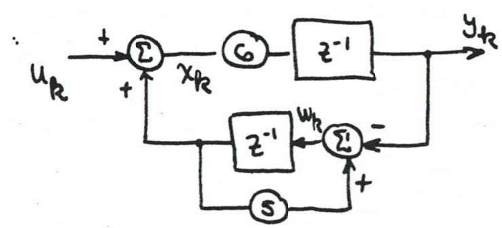
$$R_{xx}(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}$$

cf. table 4.2 #11

$$R_{xx}(z) = \frac{4}{3} \cdot \frac{1 - 1/4}{(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})}$$

$$r_{xx}(k) = \frac{4}{3}(1/2)^{|k|},\ \text{all } k$$

4.18 Redraw:



4.18 (continued)

$$Y(z) = 6z^{-1}X(z)$$
$$X(z) = U(z) + z^{-1}W(z)$$
$$W(z) = -Y(z) + 5z^{-1}W(z)$$

ie $W(z) = \dfrac{-Y(z)}{1-5z^{-1}}$

thus $X(z) = U(z) - \dfrac{z^{-1}Y(z)}{1-5z^{-1}}$

and $Y(z) = 6z^{-1}U(z) - \dfrac{6z^{-2}Y(z)}{1-5z^{-1}}$

ie $Y(z)\left[1 + \dfrac{6z^{-2}}{1-5z^{-1}}\right] = 6z^{-1}U(z)$

and $Y(z) = \dfrac{6z^{-1}(1-5z^{-1})}{1-5z^{-1}+6z^{-2}}$

Thus $H(z) = \dfrac{6z-30}{z^2-5z+6} = \dfrac{9z}{z-2} - \dfrac{4z}{z-3} - 5$

and $h_k = 9\cdot 2^k - 4\cdot 3^k - 5\delta_k, \quad k \geq 0$
$\quad = 9\cdot 2^k - 4\cdot 3^k, \quad k \geq 1$

alternate method: use state variables:

$D = 0$

$h_k = CA^{k-1}B, \quad k \geq 1$



4.19 (a) $H(z) = \mathcal{Z}\{(1/2)^k, k \geq 0\} = \dfrac{1}{1-\frac{1}{2}z^{-1}}$

$U(z) = \mathcal{Z}\{u_k\} = \dfrac{1}{1-z^{-1}}$

$\therefore W(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \dfrac{z^2}{(z-\frac{1}{2})(z-1)}$
$\quad = \dfrac{-z}{z-1/2} + \dfrac{2z}{z-1}$

thus $w_k = 2 - (1/2)^k, \quad k \geq 0$

(b) $Y(z) = U(z)\cdot H^2(z)$
$\quad = \dfrac{z^3}{(z-1)(z-1/2)^2} = \dfrac{4z}{z-1} - \dfrac{2z}{z-\frac{1}{2}} - \dfrac{z^2}{(z-1/2)^2}$

thus $y_k = 4 - 2\cdot(1/2)^k - (k+1)(1/2)^k, \quad k \geq 0$
$\quad = 4 - (k+3)(1/2)^k, \quad k \geq 0$

4.20 $H(z) = 1 + z^{-1} + z^{-2}$

(a) $Y(z) = H(z)\,U(z)$
$\quad = (1 + z^{-1} + z^{-2})\,U(z)$

thus $y_k = u_k + u_{k-1} + u_{k-2}$ (also directly from diagram)
$\quad = \delta_k + \frac{3}{2}\delta_{k-1} + \{7\cdot 2^{-k}\}_2^\infty$
$\quad = -6\delta_k - 2\delta_{k-1} + 7\cdot 2^{-k}, \quad k \geq 0$

4.20 (continued)

$$[H(z)]^2 = (1+z^{-1}+z^{-2})^2$$
$$= 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

Thus $Y(z) = (1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4})\,U(z)$

and $y_k = u_k + 2u_{k-1} + 3u_{k-2} + 2u_{k-3} + u_{k-4}$

$$= \left\{ \underset{\substack{\uparrow \\ k=0}}{1},\ \frac{5}{2},\ \frac{17}{4},\ \frac{33}{8},\ \frac{49}{16},\ \frac{49}{32}, \dots \frac{49}{2^k} \dots \right\}$$

4.21 From the problem statement,

$$p(n+1) + \frac{\alpha k}{1-\alpha k}\, p(n) = \frac{k}{1-\alpha k}\, r(0)$$

Taking unilateral transforms (to incorporate the initial condition),

$$z\left[P_u(z) - p(0)\right] + \frac{\alpha k}{1-\alpha k} P(z) = \frac{k\, r(0)}{1-\alpha k} \cdot \frac{1}{1-z^{-1}}$$

ie

$$P_u(z) = \frac{b z^{-1}}{(1-z^{-1})(1+az^{-1})} + \frac{p(0)}{1+az^{-1}}$$

where $a = \frac{\alpha k}{1-\alpha k}$ and $b = \frac{k\,r(0)}{1-\alpha k}$

$$= \frac{k\,r(0)}{1-z^{-1}} + \frac{-k\,r(0)}{1+az^{-1}} + \frac{p(0)}{1+az^{-1}}$$

from which

$$p(n) = k\,r(0) + \left[p(0) - k\,r(0)\right]\left(\frac{\alpha k}{\alpha k - 1}\right)^n, \quad n \geq 0$$

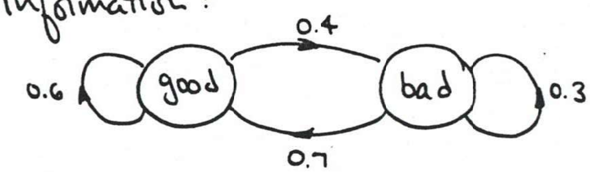


4.21 (continued)

Suppose now that $\alpha k > 1$. Then the term $\left(\frac{\alpha k}{\alpha k-1}\right)^n \to \infty$ as $n \to \infty$; hence $p(n) \to \pm\infty$, depending on the sign of $p(0) - k\,r(0)$. To obtain a limiting price of $k\,r(0)$, we want $\left(\frac{\alpha k}{\alpha k-1}\right)^n \to 0$ as $n \to \infty$; thus

$$\left|\frac{\alpha k}{\alpha k - 1}\right| < 1 \Rightarrow \alpha k < 1/2$$

4.22 Let $p_k$ = probability of good fishing on $k+1$st day. The following diagram summarizes the given information:




Then $p_{k+1} = 0.6\,p_k + 0.7(1-p_k)$

ie

$$p_{k+1} + 0.1\,p_k = 0.7$$

$$z\left[P(z) - p_0\right] + 0.1\,P(z) = \frac{0.7}{1-z^{-1}}, \quad p_0 = 0$$

$$P(z) = \frac{0.7 z^{-1}}{(1-z^{-1})(1+0.1z^{-1})} = \frac{-7/11}{1+0.1z^{-1}} + \frac{7/11}{1-z^{-1}}$$

and $p_k = \frac{7}{11}\left[1 - (-0.1)^k\right]$

thus $p_1 = 0.7$; $p_2 = \frac{7}{11}(0.99)$; $p_{10} \approx \frac{7}{11}$

4.23 Let $P_n$ = probability that the coin shows a head on the $n^{th}$ toss

Then
$$P_{n+1} = \underset{\substack{\uparrow \\ \text{prob. of} \\ \text{heads at } n}}{P_n} \cdot p + \underset{\substack{\uparrow \\ \text{prob of} \\ \text{tails at } n}}{\underbrace{(1-P_n)}}(1-p)$$

ie $$P_{n+1} + (1-2p)P_n = 1-p\ , \quad \text{with } P_0 = p'$$

Let $a = 1-2p$ and $b = 1-p$

Then $$P_{n+1} + aP_n = b$$

Taking Z - transforms,
$$z \cdot \left[P(z) - P_0\right] + aP(z) = \frac{b}{1-z^{-1}}$$

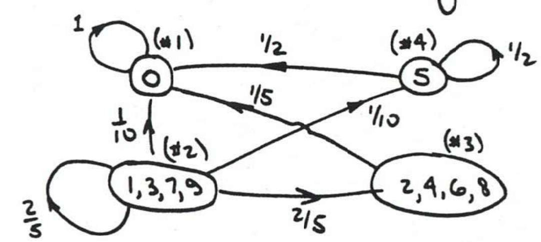
ie $$P(z) = \frac{bz^{-1}}{(1-z^{-1})(1+az^{-1})} + \frac{p'}{1+az^{-1}}$$
$$= \frac{b/(1+a)}{1-z^{-1}} + \left(p' - \frac{b}{1+a}\right) \cdot \frac{1}{1+az^{-1}}$$

Therefore,
$$P_n = \frac{1-p}{2-2p} + \left(p' - \frac{1-p}{2-2p}\right)(2p-1)^n$$
$$= \frac{1}{2} + \left(p' - \frac{1}{2}\right)(2p-1)^n$$

(note that for $p = 1/2$, $P_n = 1/2$ as expected)



4.24 One can reduce the integers of interest to those in the set $\{0,1,2,\ldots 9\}$ since we are concerned with only the units digit. Starting with a 1 at time zero, we wish to find the probability of ending with a 2 in the units digit after $N$ multiplications. Consider the equivalence classes $\{0\}$, $\{5\}$, $\{1,3,7,9\}$, $\{2,4,6,8\}$. The following Markov model shows how we traverse from set to set:




define $$\underset{=}{P} = [P_{ij}]$$ where $P_{ij}$ = probability of going from class $i$ to class $j$

$$\underset{=}{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/10 & 2/5 & 2/5 & 1/10 \\ 1/5 & 0 & 4/5 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

The matrix $\underset{=}{P}$ describes how transitions between states take place as numbers are multiplied. For example, if one started in state #2 (ie $\{1,3,7,9\}$, and an integer is drawn at random

4.24 (continued)

from $\{0,1,2,\ldots 9\}$, the probability of the units digit of the product being in state (#1), (#2), (#3), and (#4) is

$$\underset{\substack{\uparrow \\ \text{start in \#2}}}{[0\ 1\ 0\ 0]}\ \underline{\underline{P}} = \begin{bmatrix} 1/10 \\ 2/5 \\ 2/5 \\ 1/10 \end{bmatrix} \leftarrow \text{probabilities of being in the various states}$$

We wish to determine the probability of being in state #3 ($\{2,4,6,8\}$) after N transitions. We then take $\frac{1}{4}$ of this number to determine the probability of obtaining a 2 in the units digit, since within a class, all digits are equally likely. Thus our answer is

$$\frac{1}{4}\underset{\text{start with a 1}}{[0\,1\,0\,0]}\, P^N \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{end in state \#3}$$

$$= \frac{1}{4} p_{23}^{(N)}$$

Following the approach of Example 4.17, we write

$$[I - z^{-1}P]^{-1} = \begin{bmatrix} 1-z^{-1} & 0 & 0 & 0 \\ -.1z^{-1} & 1-.4z^{-1} & -.4z^{-1} & -.1z^{-1} \\ -.1z^{-1} & 0 & 1-.8z^{-1} & 0 \\ -.5z^{-1} & 0 & 0 & 1-.5z^{-1} \end{bmatrix}^{-1}$$



4.24 (continued)

$$[I - z^{-1}P]^{-1} = \frac{1}{(1-z^{-1})(1-.4z^{-1})(1-.8z^{-1})(1-.5z^{-1})} \begin{bmatrix} - & - & - & - \\ \vdots & & (1-z^{-1})(.4z^{-1})(1-.5z^{-1}) & \\ & & - & - \\ & & - & - \end{bmatrix}$$

(the only term we need)

$$[I - z^{-1}P]^{-1}\Big|_{23} = \frac{.4z^{-1}}{(1-.4z^{-1})(1-.8z^{-1})}$$

$$= \frac{-1}{1-.4z^{-1}} + \frac{1}{1-.8z^{-1}}$$

$$\therefore p_{23}^{(N)} = -\left(\frac{2}{5}\right)^N + \left(\frac{4}{5}\right)^N$$

and the probability of a 2 in the units digit is

$$\frac{1}{4} p_{23}^{(N)} = \frac{1}{4}\left[\left(\frac{4}{5}\right)^N - \left(\frac{2}{5}\right)^N\right]$$

4.25 $\Pr\{x = k\} = p^{k-1}q, \quad k = 1, 2, \ldots$

Then $P(z) \triangleq Z\{\Pr\{x=k\}\} = \sum_{k=1}^{\infty} p^{k-1} q z^{-k}$

Thus

$$\frac{d}{dz}P(z) = \sum(-k)p^{k-1}q z^{-k-1}$$

and therefore

$$\frac{d}{dz}P(z)\Big|_{z=1} = -\sum_{k=1}^{\infty} k p^{k-1} q = -\sum k\Pr\{x=k\} = -E\{x\}$$

Hence

$$E\{x\} = -\frac{d}{dz}P(z)\Big|_{z=1} = -\frac{d}{dz}\frac{q}{z-p}\Big|_{z=1}$$

$$= \frac{q}{(z-p)^2}\Big|_{z=1} = \frac{q}{(1-p)^2} = 1/q$$

4.26 Continuing from the previous problem,

$$\frac{d}{dz}P(z) = -\sum k\, p^{k-1} q\, z^{-k-1}$$

$$\frac{d^2}{dz^2}P(z) = -\sum k(-k-1)\, p^{k-1} q\, z^{-k-2}$$

and
$$\left.\frac{d^2}{dz^2}P(z)\right|_{z=1} = \sum k^2 p^{k-1} q + \sum k\, p^{k-1} q$$
$$= E\{x^2\} + E\{x\}$$

Hence
$$E\{x^2\} = \left.\frac{d^2}{dz^2}P(z)\right|_{z=1} - E\{x\}$$
$$= \left.\frac{d^2}{dz^2}P(z)\right|_{z=1} + \left.\frac{d}{dz}P(z)\right|_{z=1}$$

Since
$$\sigma^2 = E\{x^2\} - [E\{x\}]^2,$$
It follows that
$$\sigma^2 = \left.P^{(2)}(z)\right|_{z=1} + \left.P^{(1)}(z)\right|_{z=1} - \left[\left.P^{(1)}(z)\right|_{z=1}\right]^2$$
as claimed.

4.27 One method of obtaining the difference equation is to consider the reproduction for the first few months. Let $y_n$ be the number of gerbil pairs at the end of the nth month. Assume that $y_0 = 1$. We obtain the following sequence of values for $y_n$:



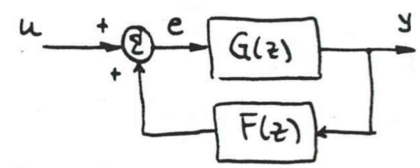
4.27 (continued)

$$\{y_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}.$$

Clearly,
$$y_n = y_{n-1} + y_{n-2} \quad \text{with } y_0 = y_1 = 1.$$
This is the same as the classical Fibonacci sequence shifted to the left one term; i.e., $y_n = f_{n+1}$, where $f_n$ was found in problem 4.5.

Thus
$$y_n = f_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \quad n \geq 0$$
and at the end of the first year,
$$y_{12} = 223.$$

4.28




Define the signal $e_k = u_k + f_k * y_k$ as shown above.

$$Y(z) = E(z)\,G(z)$$

where
$$E(z) = U(z) + F(z)\,Y(z)$$

Thus
$$Y(z) = G(z)U(z) + G(z)F(z)Y(z)$$
from which
$$Y(z) = U(z) \cdot \frac{G(z)}{1 - G(z)F(z)}$$

**4.29** (a) $G(z) = z^{-1}$

$F(z) = a$

$$H(z) = \frac{z^{-1}}{1-az^{-1}}$$

(b) $G(z) = z^{-1}$

$F(z) = -a$

$$H(z) = \frac{z^{-1}}{1+az^{-1}}$$

(c) $G(z) = 1$

$F(z) = -az^{-1}$

$$H(z) = \frac{1}{1+az^{-1}}$$

(d) inner loop: $G_1(z) = 1$

$F_1(z) = -az^{-1}$

$$H_1(z) = \frac{1}{1+az^{-1}}$$

outer loop: $G_2(z) = H_1(z) = \dfrac{1}{1+az^{-1}}$

$F_2(z) = -bz^{-1}$

$$H(z) = \frac{\frac{1}{1+az^{-1}}}{1+\frac{bz^{-1}}{1+az^{-1}}} = \frac{1}{1+(a+b)z^{-1}}$$

(e) inner loop: $G_1(z) = 1$

$F_1(z) = az^{-1}$

$$H_1(z) = \frac{1}{1-az^{-1}}$$



**4.29 (e)** (continued)

outer loop: $G_2(z) = z^{-1}H_1(z) = \dfrac{z^{-1}}{1-az^{-1}}$

$F_2(z) = -bz^{-1}$

$$H(z) = \frac{\frac{z^{-1}}{1-az^{-1}}}{1+\frac{bz^{-2}}{1-az^{-1}}} = \frac{z^{-1}}{1-az^{-1}+bz^{-2}}$$

(f) $G(z) = \dfrac{1}{1-az^{-1}} \cdot \dfrac{1}{1+bz^{-1}}$

$F(z) = c$

$$H(z) = \frac{\frac{1}{(1-az^{-1})(1+bz^{-1})}}{1-\frac{c}{(1-az^{-1})(1+bz^{-1})}} = \frac{1}{(1-c)+(b-a)z^{-1}-abz^{-2}}$$

(g) $G(z) = 1$

$$F(z) = \frac{1}{1+az^{-1}} \cdot \frac{1}{1+bz^{-1}} \cdot (-c) = \frac{-c}{(1+az^{-1})(1+bz^{-1})}$$

$$H(z) = \frac{1}{1+\frac{c}{(1+az^{-1})(1+bz^{-1})}}$$

$$= \frac{1+(a+b)z^{-1}+abz^{-2}}{(1+c)+(a+b)z^{-1}+abz^{-2}}$$

4.29 (continued)

(h) innermost loop: $H_1(z) = \dfrac{1}{1+az^{-1}}$

2nd loop: $G_2(z) = z^{-1}H_1(z) = \dfrac{z^{-1}}{1+az^{-1}}$

$F_2(z) = bz^{-1}$

$$H_2(z) = \frac{\frac{z^{-1}}{1+az^{-1}}}{1 - \frac{bz^{-2}}{1+az^{-1}}} = \frac{z^{-1}}{1+az^{-1}-bz^{-2}}$$

outermost loop: $G_3(z) = z^{-1}H_2(z) = \dfrac{z^{-2}}{1+az^{-1}-bz^{-2}}$

$F_3(z) = -cz^{-1}$

$$H(z) = \frac{\frac{z^{-2}}{1+az^{-1}-bz^{-2}}}{1+\frac{cz^{-3}}{1+az^{-1}-bz^{-2}}} = \frac{z^{-2}}{1+az^{-1}-bz^{-2}+cz^{-3}}$$

4.30 (a) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{z^{-1}}{1-az^{-1}}$

$Y(z)(1-az^{-1}) = z^{-1}U(z)$

$Y(z) - az^{-1}Y(z) = z^{-1}U(z)$

$y_k - ay_{k-1} = u_{k-1}$

alt: $y_{k+1} - ay_k = u_k$



4.30 (continued)

(b) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{z^{-1}}{1+az^{-1}}$

$Y(z)(1+az^{-1}) = z^{-1}U(z)$

$y_k + ay_{k-1} = u_{k-1}$

(c) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{1}{1+az^{-1}}$

$Y(z)(1+az^{-1}) = U(z)$

$y_k + ay_{k-1} = u_k$

(d) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{1}{1+(a+b)z^{-1}}$

$Y(z)(1+(a+b)z^{-1}) = U(z)$

$y_k + (a+b)y_{k-1} = u_k$

(e) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{z^{-1}}{1-az^{-1}+bz^{-2}}$

$Y(z)(1-az^{-1}+bz^{-2}) = z^{-1}U(z)$

$y_k - ay_{k-1} + by_{k-2} = u_{k-1}$

(f) $H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{1}{(1-c)+(b-a)z^{-1}-abz^{-2}}$

$(1-c)y_k + (b-a)y_{k-1} - aby_{k-2} = u_k$

4.30 (continued)

(g) $H(z) = \frac{Y(z)}{U(z)} = \frac{1+(a+b)z^{-1}+abz^{-2}}{(1+c)+(a+b)z^{-1}+abz^{-2}}$

$$(1+c)y_k+(a+b)y_{k-1}+aby_{k-2} = u_k+(a+b)u_{k-1}+abu_{k-2}$$

(h) $H(z) = \frac{Y(z)}{U(z)} = \frac{z^{-2}}{1+az^{-1}-bz^{-2}+cz^{-3}}$

$$y_k+ay_{k-1}-by_{k-2}+cy_{k-3} = u_{k-2}$$

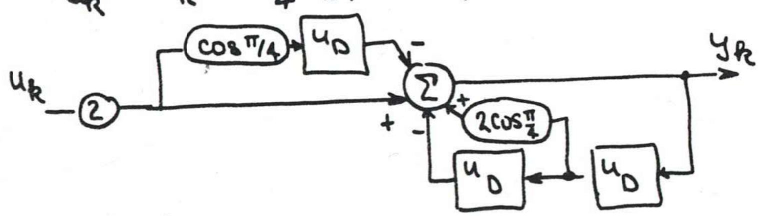
4.31 $h(n) = 2\cos\left(\frac{\pi n}{4}\right), \quad n \geq 0$

$$H(z) = \frac{2(1-z^{-1}\cos\pi/4)}{1-2z^{-1}\cos \quad +z^{-2}} \equiv \frac{Y(z)}{U(z)}$$

$$Y(z)(1-2z^{-1}\cos\pi/4+z^{-2}) = (2-2z^{-1}\cos\pi/4)U(z)$$

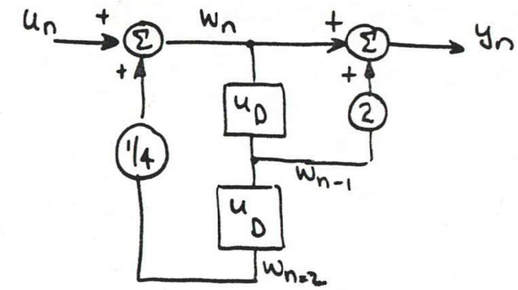
$$Y(z) = 2U(z)-2z^{-1}\cos\tfrac{\pi}{4}U(z)+2z^{-1}\cos\tfrac{\pi}{4}Y(z)-z^{-2}Y(z)$$

$$y_k = 2u_k-2\cos\tfrac{\pi}{4}u_{k-1}+2\cos\tfrac{\pi}{4}y_{k-1}-y_{k-2}$$






4.32




$$w_n = u_n + \tfrac{1}{4}w_{n-2}$$

ie $W(z)(1-\tfrac{1}{4}z^{-2}) = U(z)$

also, $y_n = w_n + 2w_{n-1}$

ie $Y(z) = W(z)(1+2z^{-1}) = \left[\frac{U(z)}{1-\frac{1}{4}z^{-2}}\right]\cdot(1+2z^{-1})$

$$= U(z)\cdot\frac{1+2z^{-1}}{1-\frac{1}{4}z^{-2}}$$

hence $H(z) = \frac{Y(z)}{U(z)} = \frac{1+2z^{-1}}{1-\frac{1}{4}z^{-2}}$

$$= \frac{5/2}{1-\frac{1}{2}z^{-1}} - \frac{3/2}{1+\frac{1}{2}z^{-1}}$$

and $h_n = \frac{5}{2}\left(\frac{1}{2}\right)^k - \frac{3}{2}\left(-\frac{1}{2}\right)^k, \quad k \geq 0$

4.33 (a) following the approach of #4.29,

$$G(z) = \frac{z^{-1}}{1-z^{-1}}$$

$$F(z) = K$$

$$H(z) = \frac{\frac{z^{-1}}{1-z^{-1}}}{1 - \frac{Kz^{-1}}{1-z^{-1}}} = \frac{z^{-1}}{1-(1+K)z^{-1}}$$

Note that the system has a pole at $z = 1+K$. For stability, this pole must be inside the unit circle; thus

$$|1+K| < 1$$

ie $\quad -1 < (K+1) < 1$

or $\quad -2 < K < 0$

(b) generalizing the result from #4.32, we can write the transfer function by inspection as

$$H(z) = \frac{1 + \frac{1}{2}z^{-1} + z^{-2}}{1 - Kz^{-1} - \frac{9}{16}z^{-2}}$$

The system has poles at $z^2 - Kz - \frac{9}{16} = 0$;

ie for $\quad P_{1,2} = \frac{K}{2} \pm \sqrt{\frac{K^2}{4} + \frac{9}{16}}$



4.33 (b) (continued)

Thus

$$-1 < \frac{K}{2} - \sqrt{\frac{K^2}{4} + \frac{9}{16}} < \frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{9}{16}} < 1$$

(since $\sqrt{\ } \geq 0$)

ie (i) $\quad -2 < K - \sqrt{K^2 + 9/4}$

$$K^2 + \frac{9}{4} < (K+2)^2 = K^2 + 4K + 4$$

$$-\frac{7}{4} < 4K \Rightarrow K > -7/16$$

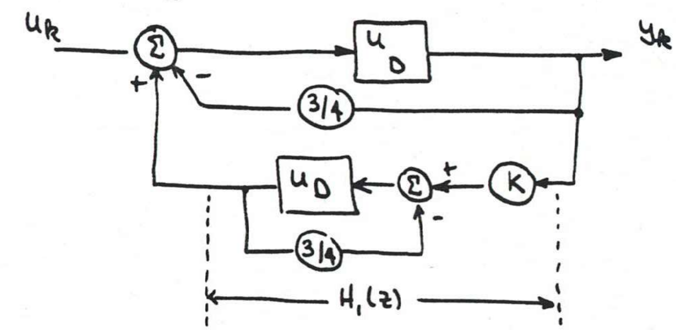
(ii) $\quad K + \sqrt{K^2 + 9/4} < 2$

$$K^2 + 9/4 < (2-K)^2 = 4 - 4K + K^2$$

$$-\frac{7}{4} < -4K \Rightarrow K < 7/16$$

Combining, the range of stability is $|K| < 7/16$

(c) To obtain the transfer function, redraw the diagram as



4.33 (c) (continued)

with $H_1(z) = \dfrac{Kz^{-1}}{1+\frac{3}{4}z^{-1}}$

and $F(z) = -\frac{3}{4} + H_1(z) = \dfrac{-\frac{3}{4}+(K-\frac{9}{16})z^{-1}}{1+\frac{3}{4}z^{-1}}$

we have
$$H(z) = \frac{z^{-1}}{1+\left[\dfrac{\frac{3}{4}-(K-\frac{9}{16})z^{-1}}{1+\frac{3}{4}z^{-1}}\right]z^{-1}}$$
$$= \frac{z^{-1}+\frac{3}{4}z^{-2}}{1+\frac{3}{4}z^{-1}+\frac{3}{4}z^{-1}-(K-\frac{9}{16})z^{-2}}$$

with poles at $z^2+\frac{3}{2}z+\left(\frac{9}{16}-K\right)=0$

ie for $p_{1,2} = -\frac{3}{4} \pm \sqrt{\frac{9}{16}-\left(\frac{9}{16}-K\right)} = -\frac{3}{4} \pm \sqrt{K}$

(i) $K>0$ : $-1 < -\frac{3}{4}-\sqrt{K}$

ie $\sqrt{K} < -\frac{3}{4}+1 = 1/4 \Rightarrow K < 1/16$

(ii) $K<0$ : $\left|-\frac{3}{4} \pm j\sqrt{-K}\right| < 1$

ie $\frac{9}{16}+(-K) < 1 \Rightarrow K > \frac{9}{16}-1 = -7/16$

Thus the system will be stable for
$$-\frac{7}{16} < K < \frac{1}{16}$$





5.1 $F_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt$ with $\omega_0 = \frac{2\pi}{T}$

(a) $T=\pi$, $\omega_0 = 2$

$$F_n = \frac{1}{\pi}\int_0^\pi \frac{At}{\pi}e^{-j2nt}dt = \frac{A}{\pi^2}\left\{\frac{te^{-j2nt}}{-j2n}\Big|_0^\pi - \int_0^\pi \frac{e^{-j2nt}}{-j2n}dt\right\}, n\neq 0$$
$$= \frac{A}{\pi^2}\left\{\frac{\pi e^{-j2n\pi}}{-j2n} + \frac{e^{-j2nt}}{4n^2}\Big|_0^\pi\right\}$$
$$= \frac{A}{\pi^2}\left\{\frac{\pi e^{-j2n\pi}}{-j2n} + \frac{e^{-j2n\pi}-1}{4n^2}\right\} = j\frac{A}{2\pi n} \quad \text{since } e^{-j2\pi n} = 1.$$

For $n=0$, $F_0 = \frac{1}{\pi}\int_0^\pi \frac{At}{\pi}dt = \frac{At^2}{2\pi^2}\Big|_0^\pi = \frac{A}{2}$.

Thus
$$F_n = \begin{cases} A/2, & n=0 \\ j\dfrac{A}{2\pi n}, & n\neq 0 \end{cases}$$

(b) $T=2\pi$, $\omega_0 = 1$

$$F_n = \frac{1}{2\pi}\int_0^\pi \sin t\, e^{-jnt}dt = \frac{1}{2\pi}\int_0^\pi \left(\frac{e^{jt}-e^{-jt}}{2j}\right)e^{-jnt}dt$$
$$= \frac{1}{j4\pi}\left\{\int_0^\pi e^{-j(n-1)t}dt - \int_0^\pi e^{-j(n+1)t}dt\right\}$$
$$= \frac{1}{j4\pi}\left\{\frac{e^{-j(n-1)t}}{-j(n-1)}\Big|_0^\pi - \frac{e^{-j(n+1)t}}{-j(n+1)}\Big|_0^\pi\right\}, \quad n\neq 1$$
$$= \frac{1}{4\pi}\left\{\frac{e^{-j(n-1)\pi}-1}{n-1} - \frac{e^{-j(n+1)\pi}-1}{n+1}\right\}, \quad n\neq 1$$

If $n$ is odd, $n\pm 1$ is even and $e^{-j(n\pm 1)} = 1$. In this case $F_n = 0$, $n$ odd & $n \neq \pm 1$. For $n$ even, $n\pm 1$ is odd. In