finally, from the top equation,

$$(10/9)\hat{h}_0 - (1/3)\hat{h}_1 = 820.125\,\hat{h}_6 = 1$$

$$\Rightarrow \hat{h}_6 = 0.001219$$

Thus

$$\{\hat{h}_k\} = \{1, 0.3333, 0.1111, 0.0370, 0.0123, 0.0041, 0.0012\}$$

A more elegant solution would proceed as

$$\begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & \beta & \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & \beta & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & \beta & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & \beta & \alpha \end{bmatrix} \begin{bmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \\ \hat{h}_4 \\ \hat{h}_5 \\ \hat{h}_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \alpha &= 1+(1/3)^2 \\ \beta &= -1/3 \end{aligned}$$

[cf (2.8.13)]

hence $\quad \underline{\hat{h}} = [\quad]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$= 1^{st} \text{ column of } \begin{bmatrix} \alpha & \beta & 0 & \cdots \\ \beta & \alpha & \beta & \cdots \\ \cdots & & \beta\ \alpha & \beta \\ 0 & 0\ 0 & 0 & \beta\ \alpha \end{bmatrix}^{-1}$$



# Chapter 3

3.1 (a) $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0 \rightarrow r_1 = -1,\ r_2 = -2$

$y(t) = c_1 e^{-t} + c_2 e^{-2t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1 - 2c_2 = 0 \end{aligned}\right\} \begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

$y(t) = 2e^{-t} - e^{-2t}$

(b) $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \longrightarrow r_1 = r_2 = -1$

$y(t) = c_1 e^{-t} + c_2 t e^{-t}$

$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1 + c_2 = 0 \end{aligned}\right\} \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

$y(t) = (1+t)e^{-t}$

(c) $r^3 + 4r^2 + 5r + 2 = 0$

$(r+2)(r+1)^2 = 0 \rightarrow r_1 = -2,\ r_2 = r_3 = -1$

$y(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 t e^{-t}$

3.1 (c) (continued)

$$\left.\begin{aligned} y(0) &= C_1 + C_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -2C_1 - C_2 + C_3 = 0 \\ \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} &= 4C_1 + C_2 - 2C_3 = 0 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 1 \\ C_2 &= 0 \\ C_3 &= 2 \end{aligned}$$

$$y(t) = e^{-2t} + te^{-t}$$

(d) $r^2 - 1 = 0$

$(r-1)(r+1) = 0 \rightarrow r_1 = 1, \quad r_2 = -1$

$y(t) = C_1e^{t} + C_2e^{-t}$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= C_1 - C_2 = 1 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 1 \\ C_2 &= 0 \end{aligned}$$

(e) $r^2 + 1 = 0$

$(r+j)(r-j) = 0 \rightarrow r_1 = -j, \quad r_2 = j$

$y(t) = C_1\cos t + C_2\sin t$

$y(0) = C_1 = 1$

$\left.\frac{dy(t)}{dt}\right|_{t=0} = C_2 = 0$

$y(t) = \cos t$



3.1 (continued)

(f) as in (e), $y(t) = C_1\cos t + C_2\sin t$

$y(0) = C_1 = 0$

$\left.\frac{dy(t)}{dt}\right|_{t=0} = C_2 = 1$

$y(t) = \sin t$

(g) as in (e), $y(t) = C_1\cos t + C_2\sin t$

$y(0) = C_1 = 1$

$\frac{dy(t)}{dt} = C_2 = 1$

$y(t) = \cos t + \sin t$

[note in (e)-(g) that the solutions obey superposition with respect to the initial conditions]

(h) $r^4 - 1 = 0$

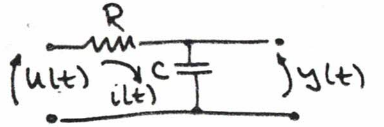
$(r-1)(r+1)(r-j)(r+j) = 0 \qquad r_1 = 1, \; r_2 = -1, \; r_3 = j, \; r_4 = -j$

$y(t) = C_1e^{t} + C_2e^{-t} + C_3\cos t + C_4\sin t$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 + C_3 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= C_1 - C_2 + C_4 = 0 \\ \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} &= C_1 + C_2 - C_3 = 0 \\ \left.\frac{d^3y(t)}{dt^3}\right|_{t=0} &= C_1 + C_2 - C_4 = 0 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 0 \\ C_2 &= 1/2 \\ C_3 &= 1/2 \\ C_4 &= 1/2 \end{aligned}$$

$$y(t) = \frac{1}{2}\left(e^{-t} + \cos t + \sin t\right)$$

3.2




The differential equation for this system is

$$y(t) = \frac{1}{C}\int^{t} i(t')dt'$$

where $u(t) = Ri(t) + \frac{1}{C}\int^{t} i(t')dt'$

ie

$$u(t) = RC\frac{dy(t)}{dt} + y(t)$$

from which

$$RC\frac{dy(t)}{dt} + y(t) = u(t)$$

For $u(t) \equiv 0$, we have

$$RC\frac{dy(t)}{dt} + y(t) = 0$$

$$RC(r) + 1 = 0 \rightarrow r = -1/RC$$

Thus $y(t) = ce^{-t/RC}$

with $y(0) = c = 2$

Hence, $y(t) = 2e^{-t/RC}$ as given

3.3 (a) $D-a$ (b) $(D-a)^3$

(c) $D^2+\omega^2$ (d) $D^2+\omega^2$

(e) $D^2+\omega^2$ (f) $(D-a)(D-b)(D-c)$

(g) $D^4(D^2+1)$



3.4 (a) $(D^2+aD+b)\,y(t) = ce^{kt}$

ie $(D-k)(D^2+aD+b)\,y(t) = 0$ with $L_A = D-k$

The characteristic equation is

$$(r-k)(r^2+ar+b) = 0$$

with roots $k$, $r_1$, and $r_2$ where $k \neq r_1, r_2$

The general solution is

$$y(t) = c_1e^{kt} + c_2e^{r_1t} + c_3e^{r_2t}$$

where $c_1e^{kt}$ is the forced solution.

(b) With $\sin\omega t = \frac{1}{j2}(e^{j\omega t} - e^{-j\omega t})$, we use the annihilator operator $L_A = (D-j\omega)(D+j\omega)$. If $r_{1,2} \neq \pm j\omega$, then the forced solution has the form

$$y_{forced}(t) = c_1e^{j\omega t} + c_2e^{-j\omega t}$$

$$= \hat{c}_1\cos\omega t + \hat{c}_2\sin\omega t$$

$$= c\cos(\omega t + \phi)$$

(3 equivalent forms)

(c) If $k$ is a root of the characteristic equation of the homogeneous equation, then the characteristic equation for $L_A L[y(t)] = 0$ will have repeated roots. In this case, the forced solution will be of the form $cte^{kt}$ if $k$ is a simple root of $k^2+ak+b=0$, and of the form $ct^2e^{kt}$ if $k$ is a double root of $k^2+ak+b=0$. The system is being driven at one of its natural modes

3.5 (a) $(D^4+8D^2+16)\,y(t) = -\sin t$

Using $L_A = D^2+1$, we begin with

$$(D^2+1)(D^4+8D^2+16)\,y(t) = 0$$

$$(r^2+1)(r^4+8r^2+16) = 0$$

$$(r^2+1)(r^2+4)^2 = 0 \;\rightarrow\; r = \pm j,\ \pm j2,\ \pm j2$$

Thus
$$y(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t + c_5\cos t + c_6\sin t$$

Applying $D^4+8D^2+16$ to the forced part of $y(t)$, we obtain

$$c_5\cos t - 8c_5\cos t + 16c_5\cos t + c_6\sin t - 8c_6\sin t + 16c_6\sin t \equiv -\sin t$$

from which $c_5 = 0$ and $c_6 = -1/9$, as given

(b) $(D^3-2D^2+D-2)\,y(t) = 0 \qquad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = 1$

$$r^3-2r^2+r-2 = 0$$

$$(r-2)(r^2+1) = 0 \;\rightarrow\; r = 2, \pm j$$

$$y(t) = c_1e^{2t} + c_2\cos t + c_3\sin t$$

$$\left.\begin{aligned} y(0) &= c_1+c_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= 2c_1+c_3 = 1 \\ \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} &= 4c_1-c_2 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 2/5 \\ c_2 &= 3/5 \\ c_3 &= 1/5 \end{aligned}$$

Hence $y(t) = \frac{1}{5}\left[2e^{2t} + 3\cos t + \sin t\right]$



3.5 (continued)

(c) $(D^4-D)\,y(t) = t^2$

Using $L_A = D^3$, we write

$$D^3(D^4-D)\,y(t) = 0$$

$$r^3(r^4-r) = 0 \;\rightarrow\; r = 0,0,0,0,1,-\tfrac{1}{2}\pm j\tfrac{\sqrt{3}}{2}$$

Thus
$$y(t) = c_1 + c_2e^t + c_3e^{-t/2}\cos\tfrac{\sqrt{3}}{2}t + c_4e^{-t/2}\sin\tfrac{\sqrt{3}}{2}t + c\,t + c\,t^2 + c\,t^3$$

Substituting in $(D^4-D)\,y(t) \equiv t^2$, we obtain

$$c_5(0-1) + c_6(0-2t) + c_7(0-3t^2) \equiv t^2$$

thus $c_5 = 0$, $c_6 = 0$, and $c_7 = -1/3$

This yields
$$y(t) = c_1 + c_2e^t + e^{-t/2}\left(c_3\cos\tfrac{\sqrt{3}}{2}t + c_4\sin\tfrac{\sqrt{3}}{2}t\right) - \frac{t^3}{3}$$

3.6 (a) $(D^2+3D+2)\,y(t) = 0 \qquad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 1$

$$r^2+3r+2 = 0 \;\rightarrow\; r_1 = -1,\ r_2 = -2$$

Thus $y(t) = c_1e^{-t} + c_2e^{-2t}$

$$\left.\begin{aligned} y(0) &= c_1+c_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1-2c_2 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}$$

$$y(t) = 3e^{-t} - 2e^{-2t}$$

3.6 (continued)

(b) $(D^2+3D+2)\,y(t) = e^{-t}$  $\quad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

Using $L_A = D+1$,

$$(D+1)(D^2+3D+2)\,y(t) = 0$$

$$(r+1)(r^2+3r+2) = 0 \rightarrow r = -2, -1, -1$$

$$y(t) = c_1e^{-2t} + c_2e^{-t} + c_3te^{-t}$$

Setting $(D^2+3D+2)\,y(t) = e^{-t}$,

$$c_3(-2e^{-t}+te^{-t}+3e^{-t}-3te^{-t}+2te^{-t})$$

$$= c_3(e^{-t}) = e^{-t} \rightarrow c_3 = 1$$

$$y(t) = c_1e^{-2t} + c_2e^{-t} + te^{-t}$$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -2c_1 - c_2 + 1 = 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$$y(t) = e^{-2t} + (-1+t)e^{-t}$$

(c) As in (b), $y(t) = c_1e^{-2t} + c_2e^{-t} + te^{-t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -2c_1 - c_2 + 1 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= -1 \\ c_2 &= 2 \end{aligned}$$

$$y(t) = -e^{-2t} + (2+t)e^{-t}$$

Note that $y(t)$ obeys superposition with respect to both forcing functions and initial conditions: the i.c.'s behave as a generalized input function.



3.7 (a) $(D^2+2D+1)\,y(t) = e^{-t}$  $\quad y(0) = 1, \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

$$(D+1)(D^2+2D+1)\,y(t) = 0$$

$$(r+1)(r^2+2r+1) = 0 \rightarrow r = -1, -1, -1$$

Thus

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t}$$

From $(D^2+2D+1)\,y(t) = e^{-t}$,

$$c_3e^{-t}(2-4t+t^2+4t-2t^2+t^2) = 2c_3e^{-t} = e^{-t} \rightarrow c_3 = 1/2$$

$$y(t) = c_1e^{-t} + c_2te^{-t} + \tfrac{1}{2}t^2e^{-t}$$

$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1 + c_2 = 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

$$y(t) = e^{-t}(1+t+t^2/2)$$

(b) $(D^2-1)\,y(t) = e^{-t}$  $\quad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

$$(D+1)(D^2-1)\,y(t) = 0$$

$$(r+1)(r^2-1) = 0 \rightarrow r = -1, -1, 1$$

$$y(t) = c_1e^{t} + c_2e^{-t} + c_3te^{-t}$$

$$(D^2-1)\,y(t) = c_3e^{-t}(-2+t-t) = e^{-t} \rightarrow c_3 = -1/2$$

$$y(t) = c_1e^{t} + c_2e^{-t} - \tfrac{t}{2}e^{-t}$$

3.7 (b) (continued)

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ \frac{dy(t)}{dt}\Big|_{t=0} &= c_1 - c_2 - \tfrac{1}{2} = 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1/4 \\ c_2 &= -1/4 \end{aligned}$$

$$y(t) = \frac{1}{4}\left(e^{t} - e^{-t} - 2te^{-t}\right)$$

(c) $(D^2+1)\,y(t) = \cos t \qquad y(0) = \frac{dy(t)}{dt}\Big|_{t=0} = 0$

$(D^2+1)(D^2+1)\,y(t) = 0$

$(r^2+1)^2 = 0 \quad \rightarrow \quad r = \pm j, \pm j$

$y(t) = c_1\cos t + c_2 \sin t + c_3 t\cos t + c_4 t \sin t$

$(D^2+1)\,y(t) = c_3(-2\sin t - t\cos t + t\cos t)$

$\qquad + c_4(2\cos t - t\sin t + t\sin t) \equiv \cos t$

$\qquad \rightarrow c_3 = 0; \; c_4 = 1/2$

$y(t) = c_1 \cos t + c_2 \sin t + \frac{t}{2}\sin t$

$y(0) = c_1 = 0$

$\frac{dy(t)}{dt} = c_2 = 0$

$\therefore y(t) = \frac{t}{2}\sin t$



3.8 (a) $(D^2 + 7D + 12)\,y(t) = u(t)$

$r^2 + 7r + 12 = 0 \quad \rightarrow \quad r = -3, -4$

$h(t) = c_1 e^{-3t} + c_2 e^{-4t}; \quad h(0) = 0, \; \frac{dh(t)}{dt}\Big|_{t=0} = 1$

$$\left.\begin{aligned} h(0) &= c_1 + c_2 = 0 \\ \frac{dh(t)}{dt} &= -3c_1 - 4c_2 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$h(t) = \left(e^{-3t} - e^{-4t}\right)\zeta(t)$

<u>Check</u>: $D\,h(t) = \left(-3e^{-3t} + 4e^{-4t}\right)\zeta(t) + \underbrace{\left(e^{-3t} - e^{-4t}\right)}_{=0 \text{ at } t=0}\zeta'(t)$

$D^2 h(t) = \left(9e^{-3t} - 16e^{-4t}\right)\zeta(t) + \underbrace{\left(-3e^{-3t} + 4e^{-4t}\right)}_{=1 \text{ at } t=0}\zeta'(t)$

$\therefore (D^2 + 7D + 12)\,h(t) = e^{-3t}(9 - 21 + 12)$

$\qquad + e^{-4t}(-16 + 28 - 12)$

$\qquad + \delta(t) \quad = \delta(t)$ ✓

(b) $(D^2 + 6D + 9)\,y(t) = u(t)$

$r^2 + 6r + 9 = 0 \qquad r = -3, -3$

$h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$h(0) = c_1 = 0$

$\frac{dh(t)}{dt}\Big|_{t=0} = -3c_1 + c_2 = 1 \quad \Rightarrow c_2 = 1$

3.8 (b) (continued)

$$h(t) = te^{-3t}\zeta(t)$$

<u>Check</u>: $D\,h(t) = (-3t+1)e^{-3t}\zeta(t) + \underbrace{te^{-3t}}_{=0 \text{ at } t=0}\delta(t)$

$$D^2h(t) = (-3+9t-3)e^{-3t}\zeta(t) + (1-3t)e^{-3t}\delta(t)$$
$$= (9t-6)e^{-3t}\zeta(t) + \delta(t)$$

$$\therefore (D^2+6D+9)h(t) = e^{-3t}(9t-6-18t+6+9t)\zeta(t) + \delta(t)$$
$$= \delta(t) \checkmark$$

(c) $(D^2+2D+9)\,y(t) = u(t)$

$r^2+2r+9=0 \qquad r = -1 \pm j\sqrt{8}$

$h(t) = c_1e^{-t}\cos\sqrt{8}t + c_2e^{-t}\sin\sqrt{8}t$

$h(0) = c_1 = 0$

$\left.\frac{dh(t)}{dt}\right|_{t=0} = c_2\left(-e^{-t}\sin\sqrt{8}t + \sqrt{8}e^{-t}\cos\sqrt{8}t\right)\Big|_{t=0} = \sqrt{8}c_2 = 1$

$\rightarrow c_2 = 1/\sqrt{8}$

$$h(t) = \frac{1}{\sqrt{8}}e^{-t}\sin\sqrt{8}t\,\zeta(t)$$

<u>Check</u>

$Dh(t) = \frac{1}{\sqrt{8}}\left[\left(-e^{-t}\sin\sqrt{8}t + \sqrt{8}e^{-t}\cos\sqrt{8}t\right)\zeta(t) + \left(\overset{=0}{\text{at } t=0}\right)\delta(t)\right]$

$D^2h(t) = \frac{1}{\sqrt{8}}\left[\left(e^{-t}\sin\sqrt{8}t - 2\sqrt{8}e^{-t}\cos\sqrt{8}t - 8e^{-t}\sin\sqrt{8}t\right)\zeta(t) + \left(\overset{=\sqrt{8}}{\text{at } t=0}\right)\delta(t)\right]$



3.8 (c) (continued)

thus $(D^2+2D+9)h(t) = \delta(t) + e^{-t}\cos\sqrt{8}t(-2+2)$
$+ e^{-t}\sin\sqrt{8}t(-7-2+9)\frac{1}{\sqrt{8}}$
$= \delta(t) \checkmark$

(d) $(D^3+6D^2+12D+8)\,y(t) = u(t)$

$(r^3+6r^2+12r+8) = (r+2)^3 = 0 \rightarrow r = -2,-2,-2$

$h(t) = (c_1+c_2t+c_3t^2)e^{-2t}$

$h(0) = c_1 = 0$

$\left.\frac{dh(t)}{dt}\right|_{t=0} = c_2 = 0$

$\left.\frac{d^2h(t)}{dt^2}\right|_{t=0} = 2c_3 = 1 \rightarrow c_3 = 1/2$

$$h(t) = \frac{t^2}{2}e^{-2t}\zeta(t)$$

<u>Check</u> $Dh(t) = (te^{-2t} - t^2e^{-2t})\zeta(t) + \underbrace{\frac{t^2}{2}e^{-2t}}_{=0 \text{ at } t=0}\delta(t)$

$D^2h(t) = (-4te^{-2t} + e^{-2t} + 2t^2e^{-2t})\zeta(t) + 0\cdot\delta(t)$

$D^3h(t) = (-6+12t-4t^2)e^{-2t}\zeta(t) + \left(\overset{=1}{\text{at } t=0}\right)\delta(t)$

$\therefore (D^3+6D^2+12D+8)h(t) = \delta(t) + e^{-2t}(-6+12t-4t^2$
$-24t+6+12t^2$
$+12t-12t^2+4t^2)\zeta(t)$

$= \delta(t) \checkmark$

3.8 (continued)

(e) $h(t) = (D-1)\hat{h}(t)$, where $\hat{h}(t)$ is found from part (d):

$$\hat{h}(t) = \frac{t^2}{2} e^{-2t} \zeta(t)$$

Thus

$$h(t) = \left(t - t^2 - \frac{t^2}{2}\right) e^{-2t} \zeta(t)$$

$$= \left(t - \frac{3t^2}{2}\right) e^{-2t} \zeta(t)$$

3.9 The response shown resembles that of a first order system,

$$(D + a) y(t) = u(t)$$

whose step response is given by

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) \zeta(t)$$

with the asymptotic value $\frac{1}{a} = 10 \rightarrow a = 0.1$

Thus assume $(D + 0.1) y(t) = t\,\zeta(t)$ with $y(0) = 0$

$$D^2 (D + 0.1) y(t) = 0$$

$$r^2 (r + 0.1) = 0 \quad \Rightarrow \quad r = 0, 0, -0.1$$

$$y(t) = c_1 e^{-t/10} + c_2 + c_3 t$$

$$(D + 0.1) y(t) = 0.1 c_2 + c_3 + 0.1 c_3 t = t$$



3.9 (continued)

$$\left.\begin{aligned} 0.1 c_2 + c_3 &= 0 \\ 0.1 c_3 &= 1 \end{aligned}\right\} \quad \begin{aligned} c_2 &= -100 \\ c_3 &= 10 \end{aligned}$$

thus $y(t) = c_1 e^{-t/10} - 100 + 10t$

and from $y(0) = 0$, $c_1 = 100$

thus $y(t) = 100\left(e^{-t/10} - 1\right) + 10t$, $t \geq 0$

3.10 From the word description of the problem we have the following equations:

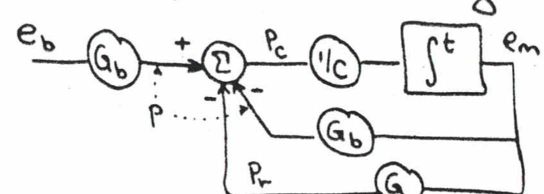
(i) $p = G_b (e_b - e_m)$

(ii) $P_r = G e_m$

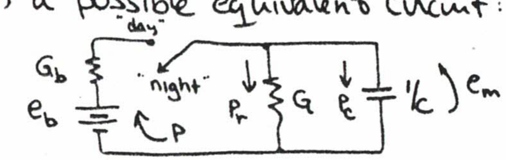
(iii) $P_c = \frac{1}{C} \frac{d e_m(t)}{dt}$

(iv) $p = P_r + P_c$

(a) a possible block diagram:




(b) a possible equivalent circuit:



3.10 (continued)

We can solve equations (i)-(iv) for $P_r(t)$ as an output in terms of the input $e_b(t)$:

$$P = P_r + P_c = P_r + \frac{1}{C}\frac{de_m(t)}{dt} = P_r + \frac{1}{GC}\frac{dP_r(t)}{dt}$$

substituting $P = G_b(e_b - e_m)$ and $e_m = \frac{P_r}{G}$,

$$G_b\left(e_b - \frac{P_r}{G}\right) = \frac{1}{GC}\frac{dP_r(t)}{dt} + P_r(t)$$

ie

$$\frac{1}{GC}\frac{dP_r(t)}{dt} + P_r(t)\left(1 + \frac{G_b}{G}\right) = G_b e_b(t)$$

This is a first-order differential equation with constant coefficients, forced by a step input. Rewriting, we have

$$\left[D + GC\left(1 + \frac{G_b}{G}\right)\right] P_r(t) = GCG_b e_b(t)$$

Which is of the form

$$(D + \alpha) P_r(t) = \beta$$

with the solution $P_r(t) = c_1 e^{-\alpha t} + c_2$,

where $(D+\alpha) c_2 = \alpha c_2 \equiv \beta \rightarrow c_2 = \beta/\alpha$

and $P_r(0) = c_1 + c_2 = 0 \rightarrow c_1 = -\beta/\alpha$

Hence $P_r(t) = \frac{GG_b}{G+G_b}\left(1 - e^{-\alpha t}\right), \quad t \geq 0$

$\alpha = GC(1 + G_b/G)$



3.11 $(D^2 + 5D + 6)\, y(t) = e^{-t}$ ; $L_A = D+1$

$(r+1)(r^2+5r+6) = 0 \rightarrow r = -1, -2, -3$

$y(t) = c_1 e^{-3t} + c_2 e^{-2t} + c_3 e^{-t}$

$(D^2+5D+6)\, y(t) = c_3(1 - 5 + 6) e^{-t} \equiv e^{-t}$

$\rightarrow c_3 = 1/2$

$y(t) = c_1 e^{-3t} + c_2 e^{-2t} + \frac{1}{2} e^{-t}$

$y(0) = c_1 + c_2 + \frac{1}{2}$

$\left.\frac{dy(t)}{dt}\right|_{t=0} = -3c_1 - 2c_2 - \frac{1}{2}$

for $c_1 = c_2 = 0$, $y(0) = 1/2$

$\left.\frac{dy(t)}{dt}\right|_{t=0} = -1/2$

Then, $y(t) = \frac{1}{2} e^{-t}$

3.12 See Figure 1.3, page 8 of the text. To apply convolution methods, we must have a relaxed system. We thus decompose the system as shown to obtain

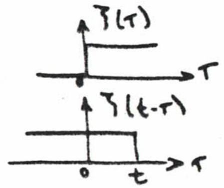
$y^{(h)}(t) = \cos t \; 3(t)$

$y^{(d)}(t) = u(t) * h(t) = \int_0^t \sin(t-\tau)\, u(\tau)\, d\tau$

Lower limit is zero since the initial conditions are specified at $t=0$

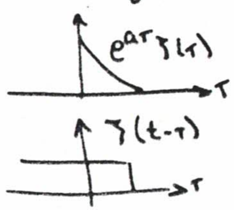
3.13 (a) $\zeta(t) * \zeta(t) = \int \zeta(\tau)\zeta(t-\tau)\,d\tau$

$$= \int_0^t 1 \cdot d\tau = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$




(b) $\zeta(t) * e^{at}\zeta(t) = \int e^{a\tau}\zeta(\tau)\cdot\zeta(t-\tau)\,d\tau$

$$= \int_0^t e^{a\tau}\,d\tau = \frac{e^{a\tau}}{a}\Big|_0^t = \begin{cases} \frac{1}{a}(e^{at}-1), & t \geq 0 \\ 0, & t < 0 \end{cases}$$




(c) $t\zeta(t) * e^{at}\zeta(t) = \int e^{a\tau}\zeta(\tau)(t-\tau)\zeta(t-\tau)\,d\tau$

$$= \int_0^t e^{a\tau}(t-\tau)\,d\tau \quad , \quad t \geq 0$$

$$= \begin{cases} \frac{1}{a^2}(e^{at}-1) - t/a, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

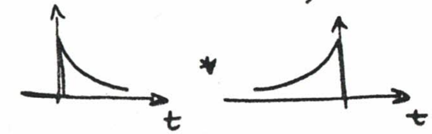
(d) $e^{at}\zeta(t) * e^{at}\zeta(t) = \int_0^t e^{a\tau}e^{a(t-\tau)}\,d\tau$

$$= \begin{cases} te^{at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



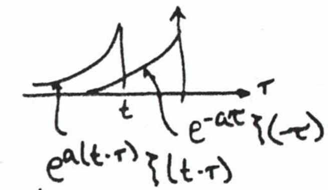
3.13 (continued)

(e) $e^{at}\zeta(t) * e^{-at}\zeta(-t)$



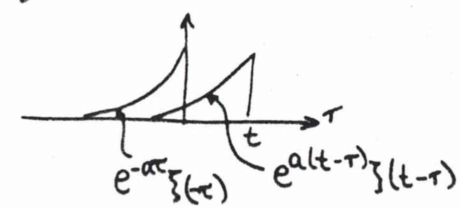

note: sketches are shown for $a<0$, since only then will the convolution integral converge.

$t<0$:




$$\int_{-\infty}^t e^{a(t-\tau)}e^{-a\tau}\,d\tau = \cdots = -\frac{e^{-at}}{2a}$$

$t>0$:




$$\int_{-\infty}^0 e^{a(t-\tau)}e^{-a\tau}\,d\tau = \cdots = -\frac{e^{at}}{2a}$$

combining, $e^{at}\zeta(t) * e^{-at}\zeta(-t) = -\frac{e^{a|t|}}{2a}$

assuming $a<0$

3.13 (continued)

(f) $(\sin t)\zeta(t) * (\sin t)\zeta(t)$

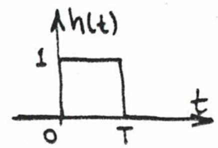
$$= \int_0^t \sin\tau \cdot \sin(t-\tau)\, d\tau$$

$$= \cdots = \begin{cases} \frac{1}{2}\sin t - \frac{1}{2}t\cos t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

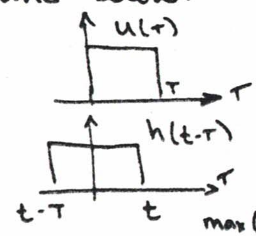
3.14 (a) The impulse response of this system may be found directly from the block diagram. An impulse input yields the output

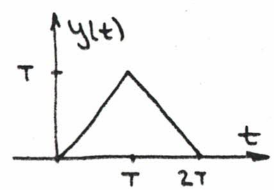
$$h(t) = \int_{-\infty}^{t} [\delta(\tau) - \delta(\tau - T)]\, d\tau = \zeta(t) - \zeta(t-T)$$

$$= \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$




The output of the system is $u(t) * h(t)$, as found below:



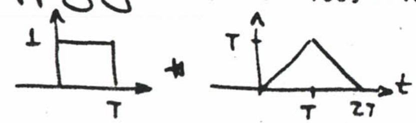




$$y(t) = u(t) * h(t) = \int_{\min(0,t-T)}^{\max(T,t)} 1 \cdot 1\, d\tau = \begin{cases} At & 0<t<T \\ A(2T-t) & T<t<2T \\ 0 & \text{otherwise} \end{cases}$$



3.14 (continued)

(b) Applying $u(t)$ to $h(t)*h(t)$, we have




Which yields

(i) $0<t<T$ $\quad y(t) = \int_0^t A\tau\, d\tau = \frac{At^2}{2}$

(ii) $T<t<2T$ $\quad y(t) = \int_{t-T}^{T} A\tau\, d\tau + \int_T^t A(2T-\tau)\, d\tau$

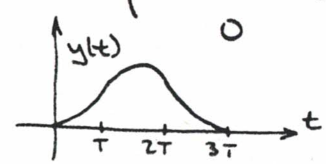
$$= 3ATt - At^2 - \frac{3AT^2}{2}$$

(iii) $2T<t<3T$ $\quad y(t) = \int_{t-T}^{2T} A(2T-\tau)\, d\tau$

$$= \frac{9}{2}AT^2 + \frac{At^2}{2} - 3AtT$$

Summarizing,

$$y(t) = \begin{cases} At^2/2 & 0<t<T \\ 3ATt - At^2 - \frac{3AT^2}{2} & T<t<2T \\ \frac{9}{2}AT^2 + \frac{At^2}{2} - 3AtT & 2T<t<3T \\ 0 & \text{elsewhere} \end{cases}$$



3.15 (a) $(D^2+7D+12)y(t) = e^t \xi(t) \qquad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = 0$

$L_A = D-1$:

$$(D-1)(D^2+7D+12)y(t) = 0$$
$$(r-1)(r^2+7r+12) = 0 \rightarrow r = -3, -4, 1$$
$$y(t) = c_1 e^t + c_2 e^{-4t} + c_3 e^{-3t}$$
$$(D^2+7D+12)y(t) = 20c_1 e^t \equiv e^t \rightarrow c_1 = 1/20$$

$$\left.\begin{aligned} y(0) &= \tfrac{1}{20} + c_2 + c_3 = 0 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= \tfrac{1}{20} - 4c_2 - 3c_3 = 0 \end{aligned}\right\} \begin{aligned} c_2 &= 1/5 \\ c_3 &= -1/4 \end{aligned}$$

$$y(t) = \tfrac{1}{20}e^t + \tfrac{1}{5}e^{-4t} - \tfrac{1}{4}e^{-3t}$$

by convolution, from 3.8a we have

$$h(t) = (e^{-3t} - e^{-4t})\xi(t);$$

thus

$$y(t) = \int_0^t (e^{-3\tau} - e^{-4\tau})e^{t-\tau}\,d\tau$$
$$= e^t \frac{e^{-4t}-1}{-4} - e^t \frac{e^{-5t}-1}{-5}$$
$$= \tfrac{1}{20}e^t - \tfrac{1}{4}e^{-3t} + \tfrac{1}{5}e^{-4t}, \quad t \geq 0$$

as above.



3.15 (continued)

(b) the first part is identical to problem 3.6 (b), from which

$$y(t) = (t-1)e^{-t} + e^{-2t}, \quad t \geq 0$$

by convolution, we have

$$r^2 + 3r + 2 = 0 \rightarrow r = -1, -2$$
$$h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

$$\left.\begin{aligned} h(0) &= c_1 + c_2 = 0 \\ \left.\frac{dh(t)}{dt}\right|_{t=0} &= -c_1 - 2c_2 = 1 \end{aligned}\right\} \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$$h(t) = (e^{-t} - e^{-2t})\xi(t)$$

Thus $y(t) = h(t) * u(t)$

$$= \int_0^t (e^{-\tau} - e^{-2\tau})e^{-(t-\tau)}\,d\tau$$
$$= e^{-t}\left\{\int_0^t 1\,d\tau - \int_0^t e^{-\tau}\,d\tau\right\}$$
$$= e^{-t}\cdot\left[t - \frac{e^{-t}-1}{-1}\right]$$
$$= te^{-t} - e^{-t} + e^{-2t}, \quad t \geq 0$$

as above.

3.16 $(D^2+3D+\frac{9}{4})\,y(t) = u(t)$

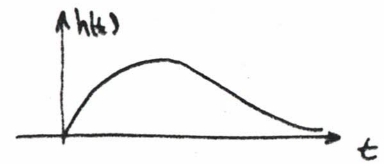
$r^2+3r+9/4 = 0 \rightarrow r = -\frac{3}{2}, -\frac{3}{2}$

$h(t) = c_1 e^{-3t/2} + c_2 t e^{-3t/2}, \quad t \geq 0$

$h(0) = c_1 = 0$

$\frac{dh(t)}{dt} = c_2 = 1$

thus $h(t) = t e^{-3t/2}$




3.17 From an inspection of the diagram,

$\frac{dy(t)}{dt} = u(t) - \int^t y(\tau)\,d\tau$

ie $(D^2+1)\,y(t) = Du(t)$

Thus $r^2+1 = 0 \rightarrow r = \pm j$

$\hat{h}(t) = c_1 \cos t + c_2 \sin t$

$\hat{h}(0) = 0 = c_1$

$\left.\frac{d\hat{h}(t)}{dt}\right|_{t=0} = c_2 = 1$



3.17 (continued)

thus $\hat{h}(t) = \sin t\,\xi(t)$

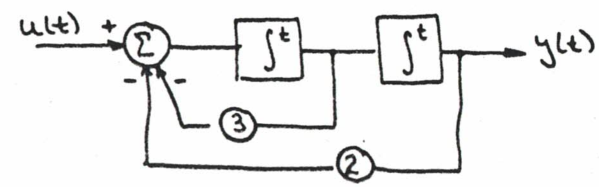
and $h(t) = D\hat{h}(t) = \cos t\,\xi(t) + \cancel{\sin t}^{\,0}\,\delta(t)$

$= \cos t\,\xi(t)$

3.18 (a) Rewrite the given equation as

$\frac{d^2y(t)}{dt^2} = u(t) - 3\frac{dy(t)}{dt} - 2y(t)$

we have:




(b) $(D^2+3D+2)\,y(t) = u(t) = e^{-t}\xi(t)$

$(D+1)(D^2+3D+2)\,y(t) = 0$

$(r+1)(r^2+3r+2) = 0 \rightarrow r = -1, -1, -2$

$y(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 t e^{-t}$

$(D^2+3D+2)\,y(t) = c_3 e^{-t}(t-2+3-3t+2t) = e^{-t}$

$\rightarrow c_3 = 1$

$\left.\begin{aligned} y(0) &= c_1+c_2 = 0 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -2c_1 - c_2 + 1 = 0 \end{aligned}\right\} \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$

3.18 (b) (continued)

Thus $y(t) = e^{-2t} + (t-1)e^{-t}, \quad t \geq 0$

(c) $r^2+3r+2=0 \longrightarrow r=-1,-2$

$h(t) = c_1 e^{-t} + c_2 e^{-2t}$

$$\left.\begin{aligned} h(0) &= c_1 + c_2 = 0 \\ \frac{dh(t)}{dt} &= -c_1 - 2c_2 = 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

$h(t) = e^{-t} - e^{-2t}, \quad t \geq 0$

(d) $y(t) = e^{-t}\xi(t) * (e^{-t} - e^{-2t})\xi(t)$

$$= \int_0^t e^{-(t-\tau)} e^{-\tau} d\tau - \int_0^t e^{-2(t-\tau)} e^{-\tau} d\tau$$

$$= e^{-t} \cdot t - e^{-2t}\left[\frac{e^{t}-1}{1}\right]$$

$= te^{-t} - e^{-t} + e^{-2t}, \quad t \geq 0$, as above.

3.19 (a) from $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{d}{dt}u(t)$

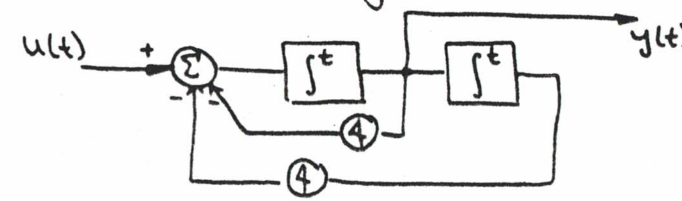
we obtain the equivalent

$$\frac{dy(t)}{dt} + 4y(t) + 4\int^t y(\tau)d\tau = u(t)$$



3.19 (a) (continued)

A possible block diagram is:




(b) $(D^2+4D+4)y(t) = Du(t) = -e^{-t}, \quad t \geq 0$

$y(0) = \frac{dy(t)}{dt} = 0$

with $L_A = D+1$,

$(D+1)(D^2+4D+4)y(t) = 0$

$(r+1)(r^2+4r+4) = 0 \rightarrow r = -1, -2, -2$

$y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 t e^{-2t}$

$(D^2+4D+4)y(t) = (c_1 - 4c_1 + 4c_1)e^{-t} = -e^{-t}$

$c_1 = -1$

$$\left.\begin{aligned} y(0) &= -1 + c_2 + 0 = 0 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= 1 - 2c_2 + c_3 = 0 \end{aligned}\right\} \quad \begin{aligned} c_2 &= 1 \\ c_3 &= 1 \end{aligned}$$

$y(t) = -e^{-t} + (1+t)e^{-2t}, \quad t \geq 0$

(c) from $(D^2+4D+4)y(t) = Du(t)$,

$$h(t) = D\hat{h}(t),$$

where

$$\hat{h}(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$\hat{h}(0) = c_1 = 0$$

$$\left.\frac{d\hat{h}(t)}{dt}\right|_{t=0} = c_2 = 1$$

$$\hat{h}(t) = te^{-2t}\zeta(t)$$

$$h(t) = (1-2t)e^{-2t}\zeta(t)$$

(d)

$$y(t) = h(t) * e^{-t}\zeta(t)$$

$$= \int_0^t (1-2\tau)e^{-2\tau}e^{-(t-\tau)}\,d\tau$$

$$= e^{-t}\left[\int_0^t e^{-\tau}d\tau - 2\int_0^t \tau e^{-\tau}d\tau\right]$$

$$= e^{-t}\left[\frac{e^{-t}-1}{-1} - 2\left(te^{-t} - e^{-t} + 1\right)\right]$$

$$= -e^{-2t} + e^{-t} + 2te^{-2t} + 2e^{-2t} - 2e^{-t}$$

$$= -e^{-t} + (1+2t)e^{-2t}, \quad t \geq 0$$

This answer does not agree with that from part (b). Note that $\left.\frac{dy(t)}{dt}\right|_{t=0} = +1$, rather than 0 as specified. This observation suggests that



3.19 (d) (continued)

the specified initial conditions do not represent those of a relaxed system (note from the diagram that for $u(t) = e^{-t}\zeta(t)$, $\left.\frac{dy(t)}{dt}\right|_{t=0} = 1$ if there is no energy storage in the system). To account for this energy storage, we must add the term $y^{(h)}(t)$ (cf. page 8 in the text, and problem 3.12), where $y^{(h)}(t)$ corresponds to a homogeneous response with appropriate initial conditions. With $y(t) = y^{(d)}(t) + y^{(h)}(t)$, $y(0) = 0$, and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$, we must choose $y^{(h)}(t)$ to satisfy

$$(D^2+4D+4)y^{(h)}(t) = 0$$

$$y^{(h)}(0) = 0$$

$$\left.\frac{dy^{(h)}(t)}{dt}\right|_{t=0} = -1$$

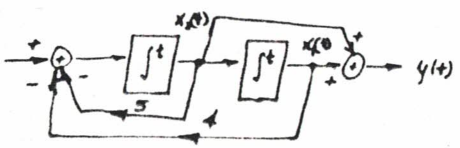
$$\rightarrow y^{(h)}(t) = -te^{-2t}\zeta(t)$$

and

$$y(t) = -e^{-t} + (1+2t)e^{-2t} + y^{(h)}(t)$$

$$= -e^{-t} + (1+t)e^{-2t}$$

(3.20)




$x_1''(t) = u(t) - 5x_1'(t) - 4x_1(t)$

$x_1''(t) + 5x_1'(t) + 4x_1(t) - u(t) = 0$

$y(t) = x_1(t) + x_1'(t)$

In operator notation we have

$$(D^2+5D+4)[x_1(t)] = u(t)$$
$$(D+1)[x_1(t)] = y(t)$$

$\therefore \quad (D^2+5D+4)(D+1)\,x_1(t) = (D+1)\,u(t) = (D^2+5D+4)\,y(t)$

$\therefore \quad (D^2+5D+4)\,y(t) = (D+1)\,u(t)$

or $\quad (D+4)(D+1)\,y(t) = (D+1)\,u(t)$

or $\quad (D+4)\,y(t) = u(t)$

$\Rightarrow \quad H(j\omega) = \dfrac{1}{j\omega+4}$

And so $\quad |H(j\omega)| = \dfrac{1}{\sqrt{16+\omega^2}}, \quad \measuredangle H(j\omega) = -\tan^{-1}(\omega/4)$

For $\quad u(t) = \sin\omega t = \cos(\pi/2 - \omega t) \Rightarrow y(t) = |H(j\omega)| \sin[\omega t + \measuredangle H(j\omega)]$

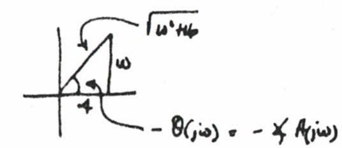
Thus $\quad y(t) = |H(j\omega)| \{ \sin\omega t \cos(\measuredangle H(j\omega)) + \sin(\measuredangle H(j\omega)) \cos\omega t \}$

From $\measuredangle H(j\omega) = -\tan^{-1}(\omega/4)$ we can obtain expressions for



(3.20 cont)

$\cos(\measuredangle H)$ and $\sin(\measuredangle H)$ as:




From the sketch:

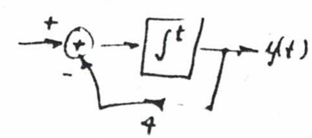
$$\cos(\measuredangle H) = \frac{4}{\sqrt{\omega^2+16}} \quad \text{and} \quad \sin(\measuredangle H) = \frac{-\omega}{\sqrt{\omega^2+16}}$$

Thus

$$y(t) = |H| \sin(\omega t + \measuredangle H)$$
$$= |H| \{ \sin\omega t \cos\measuredangle H + \cos\omega t \sin\measuredangle H \}$$
$$= \frac{1}{\sqrt{\omega^2+16}} \left\{ \sin\omega t \frac{4}{\sqrt{\omega^2+16}} + \cos\omega t \frac{-\omega}{\sqrt{\omega^2+16}} \right\}$$
$$= \left( \frac{1}{\sqrt{16+\omega^2}} \right)^2 \{ 4\sin\omega t - \omega\cos\omega t \}$$
$$= \frac{1}{16+\omega^2} \{ 4\sin\omega t - \omega\cos\omega t \}.$$

(3.21)

From Problem (3.20) we found that the differential equation for the system is $(D+4)\,y(t) = u(t)$. This has a block diagram of the form




$y'(t) + 4y(t) = u(t)$