(3.21 cont)

The impulse response of this system is given by:

$$y'(t) + 4y(t) = u(t)$$

$$r+4=0, \quad r=-4 \quad \Rightarrow h(t) = Ce^{-4t}$$

$$\& \quad h(0) = 1 = C \quad \therefore h(t) = e^{-4t}\xi(t)$$

Now $y(t) = \int_0^t u(\tau)h(t-\tau)\,d\tau = \int_0^t e^{-4(t-\tau)} \sin\omega\tau \, d\tau$

$$= e^{-4t}\int_0^t \sin\omega\tau \, e^{4\tau}\, d\tau$$

$$= e^{-4t}\left\{ e^{4\tau} \cdot \frac{4\sin\omega\tau - \omega\cos\omega\tau}{4^2+\omega^2}\Big|_0^t \right\}$$

$$= e^{-4t}\left\{ e^{4t} \cdot \frac{4\sin\omega t - \omega\cos\omega t}{16+\omega^2} - \frac{4\sin 0 + \omega\cos 0}{16+\omega^2}\right\}$$

$$= \frac{4\sin\omega t - \omega\cos\omega t}{16+\omega^2} + \frac{\omega e^{-4t}}{16+\omega^2}$$

The first term is the steady-state output. The second term is the transient response of the system. The first term is, of course, identical to the result obtained in Problem 3.20.

(3.22) The differential equation for the system is:

$$D^2y(t) + 2Dy(t) + 2y(t) = Du(t)$$

The transfer function is thus

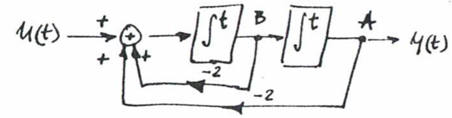
$$H(j\omega) = \frac{j\omega}{(j\omega)^2 + 2(j\omega) + 2}$$

(3.22 cont.)

To find a block diagram solve for the highest order derivative of $y(t)$. We obtain

$$D^2y(t) = Du(t) - 2Dy(t) - 2y(t)$$

If the input were $u(t)$, then a block diagram would be straight forward and given as shown below. Since the input is $Du(t)$ instead of $u(t)$ we must modify where the output is taken.



Since the input is $Du(t)$, by linearity, we must differentiate the output $y(t)$. Thus the actual block diagram is the above with the output taken from point B.

Now $|H(j\omega)|^2 = \dfrac{\omega^2}{(2-\omega^2)^2 + (2\omega)^2} = \dfrac{\omega^2}{4+\omega^4}$

and $\measuredangle H(j\omega) = \pi/2 - \measuredangle(2-\omega^2 + j2\omega) \equiv \theta(\omega)$

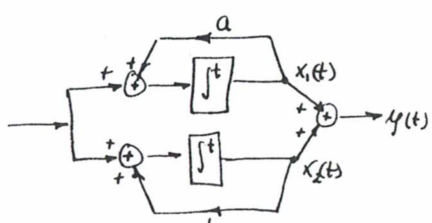
If $u(t) = \cos\omega_1 t - \sin\omega_2 t$, then

$$y(t) = |H|\cos(\omega_1 t + \measuredangle H) - |H|\sin(\omega_2 t + \measuredangle H)$$

$$= \left(\frac{\omega_1^2}{4+\omega_1^4}\right)^{1/2}\cos[\omega_1 t + \theta(\omega_1)] - \left(\frac{\omega_2^2}{4+\omega_2^4}\right)^{1/2}\sin[\omega_2 t + \theta(\omega_2)]$$

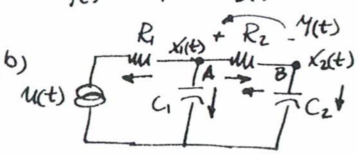
(3.23)

(a)



$$x_1'(t) = a x_1(t) + u(t)$$
$$x_2'(t) = b x_2(t) + u(t) \quad \Rightarrow \quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$y(t) = x_1(t) + x_2(t) \qquad C = [1 \;\; 1], \; D = 0.$$

(b)



$x_1(t)$ and $x_2(t)$ are the voltages across the capacitors C_1 and C_2, respectively.

Thus writing node eqs we obtain:

$$\frac{x_1(t) - u(t)}{R_1} + \frac{x_1(t) - x_2(t)}{R_2} + x_1'(t) C_1 = 0$$

$$\frac{x_2(t) - x_1(t)}{R_2} + x_2'(t) C_2 = 0$$

$$y(t) = x_1(t) - x_2(t)$$

Thus

$$x_1'(t) = \frac{1}{C_1}\left(\frac{-1}{R_1} - \frac{1}{R_2}\right) x_1(t) + \frac{1}{C_1 R_2} x_2(t) + \frac{1}{C_1 R_1} u(t)$$

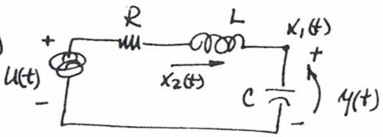
$$x_2'(t) = \frac{1}{C_2}\frac{1}{R_2} x_1(t) - \frac{1}{C_2 R_2} x_2(t)$$

$$y(t) = x_1(t) - x_2(t)$$

(3.23 cont.)

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix} \qquad C = [1 \;\; -1]$$
$$D = 0$$

(c)



$x_2(t)$ is the current through L and $x_1(t)$ is the voltage across C.

Thus we have

$$R x_2(t) + L x_2'(t) + x_1(t) = u(t) \quad \text{(loop equation)}$$

$$C x_1'(t) = x_2(t) \quad \text{(node equation)}$$

$$y(t) = x_1(t)$$

Thus $A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}, \; C = [1 \;\; 0], \; D = 0.$

(3.24) For a system with system matrix A, the system is stable iff $g(\lambda) = \det(A - \lambda I) = 0$ has roots λ_i all with real parts < 0 (negative).

(a) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow \lambda_1 = a, \; \lambda_2 = b$

$\therefore$ system stable iff $a < 0$, $b < 0$
(assuming a, b real)

(3.24 cont)

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_2R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & \frac{-1}{C_2R_2} \end{bmatrix}$$

$$g(\lambda) = \det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & b-\lambda \end{bmatrix} = (a-\lambda)(b-\lambda) + c = 0$$

where $a = \frac{-1}{C_1R_1} - \frac{1}{C_2R_2}$, $b = -\frac{1}{C_2R_2}$, $c = \frac{-1}{C_1C_2R_1R_2}$

and we know that a, b, c are all < 0.

Now $g(\lambda) = \lambda^2 - (a+b)\lambda + ab + c = 0$

Further let $ab + c = \beta$. Then $\beta = \left(\frac{-1}{C_2R_2}\right)\left(\frac{-1}{C_1R_1} - \frac{1}{C_2R_2}\right) - \frac{1}{C_1C_2R_1R_2}$

$$= \left(\frac{1}{C_2R_2}\right)^2 > 0$$

Let $-(a+b) = \alpha$; then $\alpha > 0$.

Thus $\lambda_1, \lambda_2 = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta}}{2}$

But $\beta > 0 \Rightarrow -4\beta < 0 \Rightarrow (\alpha^2 - 4\beta)^{1/2} < \alpha$

$\therefore$ System is always stable for any values of R_1, R_2, C_1, C_2 which are > 0. (This result is, of course, clear from the structure of the circuit.)

(c)

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -R/L \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -\beta & -\gamma \end{bmatrix} \text{ with } \alpha, \beta, \gamma > 0.$$

(3.24 cont)

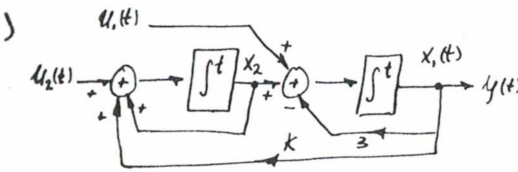
$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & \alpha \\ -\beta & -\lambda-\gamma \end{pmatrix} = \lambda^2 + \lambda\gamma + \alpha\beta = 0$$

$$\therefore \lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\alpha\beta}}{2}$$

But $\alpha\beta > 0 \Rightarrow -4\alpha\beta < 0 \Rightarrow (\gamma^2 - 4\alpha\beta)^{1/2} < \gamma$

$\therefore$ This system is always stable for any values of $R, L, C > 0$.

(3.25)



(a)

$x_1'(t) = 3x_1(t) + x_2(t) + u_1(t)$

$x_2'(t) = k\,x_1(t) + x_2(t) + u_2(t)$

$y(t) = x_1(t)$

$$A = \begin{bmatrix} 3 & 1 \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For stability the real part of the eigenvalues of A must be negative.

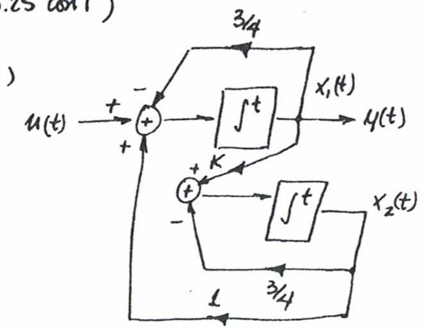
$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ k & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + (3-k) = 0$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{16 - 4(3-k)}}{2} = 2 \pm \sqrt{1+k}$$

$\left.\begin{array}{l} \lambda_1 = 2 + \sqrt{1+k} \\ \lambda_2 = 2 - \sqrt{1+k} \end{array}\right\} \Rightarrow \lambda_1$ can never have a real part < 0. System is never stable.

(3.25 cont)

(b)



$$\left.\begin{aligned} x_1'(t) &= -\tfrac{3}{4}x_1(t) + x_2(t) + u(t) \\ x_2'(t) &= K x_1(t) - \tfrac{3}{4}x_2(t) \end{aligned}\right\} \quad A = \begin{bmatrix} -3/4 & 1 \\ K & -3/4 \end{bmatrix}$$

$$g(\lambda) = \det\begin{pmatrix} -\tfrac{3}{4}-\lambda & 1 \\ K & -\tfrac{3}{4}-\lambda \end{pmatrix} = \lambda^2 + \frac{3}{2}\lambda + \frac{9}{16} - K = 0$$

$$\lambda_1, \lambda_2 = \frac{-\tfrac{3}{2} \pm \sqrt{(\tfrac{3}{2})^2 - 4(\tfrac{9}{16} - K)}}{2} = -\tfrac{3}{4} \pm \sqrt{K}$$

$$\left.\begin{aligned} \lambda_1 &= -\tfrac{3}{4} + \sqrt{K} \\ \lambda_2 &= -\tfrac{3}{4} - \sqrt{K} \end{aligned}\right\} \quad \therefore \text{ For stability } \sqrt{K} - \tfrac{3}{4} < 0$$

$$\Rightarrow K < \frac{9}{16}.$$

(3.26) In general, we have:

$$A = \sum_{i=1}^{n} \lambda_i E_i \quad \& \quad f(A) = \sum_{i=1}^{n} f(\lambda_i) E_i \quad \text{for distinct } \lambda_i.$$

Thus $f(A) = e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} E_i$

where E_i can be obtained via: $E_1 = \dfrac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$, $E_2 = \dfrac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$

(for $n = 2$)

(3.26 cont)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ $\quad g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 3/4$

Thus $E_1 = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ (Note: $E_1 + E_2 = I$)

$$\therefore\ e^{At} = e^{t/2}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} + e^{\frac{3}{4}t}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} e^{3t/4} & 0 \\ 2e^{3t/4} - 2e^{t/2} & e^{t/2} \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ \frac{1}{16} & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 3/8,\ \lambda_2 = 5/8$

$E_1 = \begin{bmatrix} 1/2 & -1 \\ -1/4 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{bmatrix}$

$$\therefore\ e^{At} = \begin{bmatrix} \frac{1}{2}(e^{3t/8} + e^{5t/8}) & e^{\frac{5t}{8}} - e^{3t/8} \\ \frac{1}{4}(e^{5t/8} - e^{3t/8}) & \frac{1}{2}(e^{3t/8} + e^{5t/8}) \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$

Thus $E_1 = \begin{bmatrix} 1/2 & 1/4 \\ -1 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & -1/4 \\ 1 & 1/2 \end{bmatrix}$

$$\therefore\ e^{At} = \begin{bmatrix} \frac{1}{2}(1 + e^t) & \frac{1}{4}(1 - e^t) \\ e^t - 1 & \frac{1}{2}(1 + e^t) \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 1/2$

In the case of repeated roots $A = \lambda E_1 + N_1$

(3.26 cont.)

and $f(A) = f(\lambda)E_1 + f'(\lambda)N_1$ (2x2 case)

$\Rightarrow e^{At} = e^{\lambda t}E_1 + te^{\lambda t}N_1$

In the 2x2 case $E_1 = I$ and $N_1 = A - \lambda I$ (see pg 90)

$\therefore \quad N_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \quad e^{At} = e^{t/2}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3/4 & -1/2 \\ -15/32 & 1/2 \end{bmatrix} \Rightarrow \lambda_1 = 1/8, \; \lambda_2 = 9/8$

$E_1 = \begin{bmatrix} 3/8 & 1/2 \\ 15/32 & 3/8 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 5/8 & -1/2 \\ -15/32 & 5/8 \end{bmatrix}$

$$e^{At} = e^{t/8}E_1 + e^{9t/8}E_2$$

$$= \begin{bmatrix} 1/8(3e^{t/8} + 5e^{9t/8}) & 1/2(e^{t/8} - e^{9t/8}) \\ 15/32(e^{t/8} - e^{9t/8}) & 1/8(3e^{t/8} + 5e^{9t/8}) \end{bmatrix}$$

(f) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_1 = 1+j, \; \lambda_2 = 1-j$

$\Rightarrow E_1 = \frac{1}{2}\begin{bmatrix} 1 & -\frac{1}{j} \\ \frac{1}{j} & 1 \end{bmatrix} \qquad E_2 = \frac{1}{2}\begin{bmatrix} 1 & +1/j \\ -1/j & 1 \end{bmatrix}$

$\therefore e^{At} = e^{(1+j)t}E_1 + e^{(1-j)t}E_2 = e^t\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

(3.26 cont)

(g) $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = -1$

$A = -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda E_1 + N_1$

$\therefore e^{At} = e^{-t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e^{-t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$

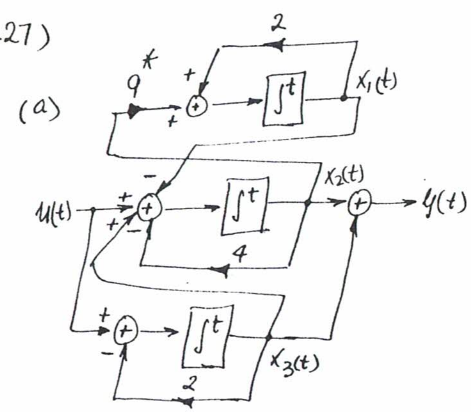
(h) $A = \begin{bmatrix} -4 & -1 \\ 16 & 4 \end{bmatrix}, \; \lambda_1 = \lambda_2 = 0$

$A = \lambda E_1 + N_1 = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 16 & 4 \end{bmatrix}$

$\therefore e^{At} = e^{0t}E_1 + te^{0t}N_1 = I + tA = \begin{bmatrix} 1-4t & -t \\ 16t & 1+4t \end{bmatrix}$

(3.27)

(a)



*Note: In the first printing the book has an "a" instead of a nine.

(3.27 cont.)

$$x_1'(t) = 2x_1(t) + 9x_2(t)$$
$$x_2'(t) = -x_1(t) - 4x_2(t) + u(t)$$
$$x_3'(t) = -2x_3(t) + u(t)$$

$$A = \begin{bmatrix} 2 & 9 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = [0 \ 1 \ 1], \quad D = 0$$

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -2, \ \lambda_2 = -1, \ \lambda_3 = -1.$$

Because we have one repeated root A is of the form:

$$A = \lambda_1 E_1 + \lambda_2 E_2 + N_2$$

There are several methods one can use to find E_1, E_2, N_2. We shall use a method which is not covered in the text but is straightforward and useful to know.

Consider the function of a matrix $f(A) = (sI - A)^{-1}$. Then we have that, from (2.85),

$$f(A) = f(\lambda_1) E_1 + f(\lambda_2) E_2 + f'(\lambda_2) N_2$$

where $f(\lambda_i) = \frac{1}{s - \lambda_i} \Rightarrow f'(\lambda_i) = \frac{1}{(s - \lambda_i)^2}$

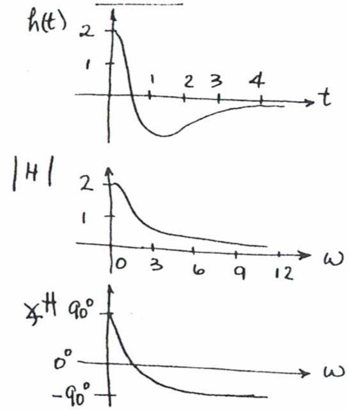
Thus

$$(sI - A)^{-1} = \frac{1}{s - \lambda_1} E_1 + \frac{1}{s - \lambda_2} E_2 + \frac{1}{(s - \lambda_2)^2} N_2$$

Now $(sI - A)^{-1} = \begin{bmatrix} s-2 & -9 & 0 \\ 1 & s+4 & 0 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}$

(3.27 cont.)

Sketches:



(b) Choose $x_1(t)$ as the output of right most integrator and $x_2(t)$ as the output of the left most integrator. Then we have

$$A = \begin{bmatrix} 0 & 1 \\ -26 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [29 \ 0], \quad D = 0$$

Again compute $(sI-A)^{-1}$ and expand in partial fractions. The "coefficients" of the partial fraction terms are matrices, E_1 & E_2.

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -1 - 5j, \ \lambda_2 = -1 + 5j$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 26 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s + 26} \begin{bmatrix} s+2 & 1 \\ -26 & s \end{bmatrix}$$

$$= \frac{1}{s - (1+5j)} \begin{bmatrix} \frac{5-j}{10} & \frac{-j}{10} \\ \frac{26j}{10} & \frac{5+j}{10} \end{bmatrix} + \frac{1}{s - (1-5j)} \begin{bmatrix} \frac{5+j}{10} & j/10 \\ \frac{-26j}{10} & \frac{5-j}{10} \end{bmatrix}$$

$$= \frac{1}{s - \lambda_1} E_1 + \frac{1}{s - \lambda_2} E_2 .$$

(3.27 cont.)

Continuing, $(sI-A)^{-1} = \frac{1}{(s+1)^2(s+2)} \begin{bmatrix} (s+2)(s+4) & 9(s+2) & 9 \\ -(s+2) & (s+2)(s-2) & s-2 \\ 0 & 0 & (s+1)^2 \end{bmatrix}$

$$= \frac{1}{s+2} \underbrace{\begin{bmatrix} 0 & 0 & 9 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} + \frac{1}{s+1} \underbrace{\begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}}_{E_2} + \frac{1}{(s+1)^2} \underbrace{\begin{bmatrix} 3 & 9 & 9 \\ -1 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}}_{N_2}$$

where E_1 and E_2 and N_2 are obtained by a partial fraction expansion, i.e.,

$$E_1 = (s+2)(sI-A)^{-1}\Big|_{s=-2}, \text{ etc.}$$

Notice: $\begin{cases} E_1+E_2 = I, & E_1E_2 = 0, \quad E_2^2 = E_2, \quad E_1^2 = E_1 \\ E_1N_2 = 0, & E_2N_2 = 0, \quad N_2^2 = 0 \end{cases}$

Then: $A = \lambda_1 E_1 + \lambda_2 E_2 + N_2 = -2\begin{bmatrix} 0 & 0 & 9 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 9 \\ -1 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

And so $e^{At} = e^{-2t}E_1 + e^{-t}E_2 + te^{-t}N_2$

$$= \begin{bmatrix} (1+3t)e^{-t} & 9te^{-t} & 9e^{-2t} + 9(t-1)e^{-t} \\ -te^{-t} & (1-3t)e^{-t} & -4e^{-2t} + (4-3t)e^{-t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

(3.27 cont)

(c) see above

(d) $H(j\omega) = C(j\omega I - A)^{-1}B$

Now $CE_1B = -3$, $CE_2B = 5$, $CN_2B = -6$

Hence $H(j\omega) = \frac{-3}{j\omega+2} + \frac{5}{j\omega+1} + \frac{-6}{(j\omega+1)^2}$

$$= \frac{2(j\omega-1)(j\omega+5/2)}{(j\omega+2)(j\omega+1)^2}$$

Frequency Response: Note: $\left|\frac{j\omega-1}{j\omega+1}\right| = 1$

Thus $|H(j\omega)| = 2\left|\frac{j\omega+5/2}{(j\omega+2)(j\omega+1)}\right|$

$$= 2\sqrt{\frac{\omega^2+25/4}{(\omega^2+4)(\omega^2+1)}}$$

The factor $\left(\frac{j\omega-1}{j\omega+1}\right)$ will contribute to the phase response.

Impulse response:

$$h(t) = CE_1B\,e^{-2t} + (CE_2B + CE_3Bt)e^{-t}$$

$$= \begin{cases} -3e^{-2t} + (5-6t)e^{-t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

(3.27 cont)

Note: $E_1 = E_2^*$, $E_1 + E_2 = I$, $E_1 E_2 = 0$

Thus $e^{At} = e^{\lambda_1 t}E_1 + e^{\lambda_2 t}E_2 = e^{-t}\begin{bmatrix} \cos 5t + \frac{\sin 5t}{5} & \frac{\sin 5t}{5} \\ -\frac{26 \sin 5t}{5} & \cos 5t - \frac{\sin 5t}{5} \end{bmatrix}$

Impulse Response:

$$h(t) = Ce^{At}B = [29 \ \ 0]\, e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{29 e^{-t}\sin 5t}{5} \ \xi(t)$$

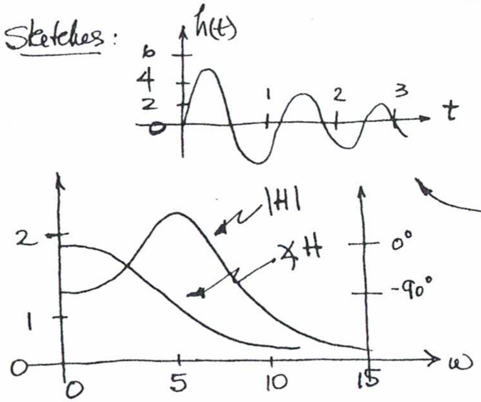
Transfer function:

$$H(j\omega) = D + C(j\omega I - A)^{-1}B = \frac{29}{(j\omega)^2 + 2j\omega + 26}$$

$\begin{cases} \text{poles: } 1 \pm 5j \\ \text{zeros: } 2 \text{ at } \infty \end{cases}$

Frequency response: $|H(j\omega)| = \dfrac{29}{\sqrt{26-\omega^2+4\omega^2}}$

Sketches:



t	$\frac{29}{5}e^{-t}\sin 5t$
.1	2.52
.2	3.99
.3	4.28
.4	3.54
.5	2.11
.7	-1.01
1.0	-2.05
1.2	-.488
1.4	.94
1.6	1.16

etc.

(3.28) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [c_1, c_2]$ $D = 0$

Note: In the first printing there is an error in the sign of the a_{22} entry in A. It should be -3, not 3. If 3 is used the eigen values are 1 and 2 implying the system is unstable. If -3 is used, we obtain

$$g(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 3\lambda + \lambda^2 + 2 = 0$$

$\Rightarrow \lambda_1 = -1$, $\lambda_2 = -2$.

Then $A = -E_1 - 2E_2$ where we can obtain E_1 & E_2 via

$$(sI-A)^{-1} = \frac{1}{s+1}E_1 + \frac{1}{s+2}E_2 = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

or $E_1 = I - E_2$ and $E_2 = -(I+A) = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow E_1 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$. Thus

$$e^{At} = e^{-t}\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

Now $H(j\omega) = D + C(j\omega I - A)^{-1}B$

$$= D + [c_1 \ \ c_2]\begin{bmatrix} 1 \\ j\omega \end{bmatrix}\frac{1}{(j\omega+1)(j\omega+2)}$$

$$= \frac{d((j\omega)^2 + 3j\omega + 2) + c_1 + c_2 j\omega}{(j\omega+1)(j\omega+2)} \qquad (1)$$

(3.28 cont.)

From the graph we have a DC gain of unity and two zeros at $\omega = \pm 1$. Thus

$$H(j\omega) = \frac{2((j\omega)^2+1)}{(j\omega+1)(j\omega+2)} \qquad (2)$$

At $j\omega=0$, $H(0)=1$ giving the DC gain. The term $(j\omega)^2+1$ gives us the zeros at $\omega = \pm 1$. Equating coefficients of like powers of $(j\omega)$ in (1) and (2) in the numerators gives:

$$\left.\begin{array}{ll} (j\omega)^0 : & 2d + c_1 = 2 \\ (j\omega)^1 : & 3d + c_2 = 0 \\ (j\omega)^2 : & d = 2 \end{array}\right\} \Rightarrow \begin{array}{l} d = 2 \\ c_1 = -2 \\ c_2 = -6 \end{array}$$

(3.29) There are (at least) three possible approaches:

(a) Use state variable methods or classical methods to solve for $w(t)$ in

$$b_n \frac{d^n}{dt^n} w(t) + \cdots + b_1 \frac{dw(t)}{dt} + b_0 w(t) = u(t)$$

Then operate on $w(t)$ to obtain $y(t)$ using superposition

$$y(t) = L_D[w(t)] = [a_0 + a_1 D + \cdots + a_m D^m] w(t)$$

$$= a_0 w(t) + a_1 \frac{d}{dt} w(t) + \cdots + a_m \frac{d^m}{dt^m} w(t)$$

(3.29 cont.)

(b) If $m \leq n$, we can integrate both sides of the given equation n times to obtain an equation for $y(t)$ in terms of integrals of $y(t)$ and $u(t)$, with no derivatives present:

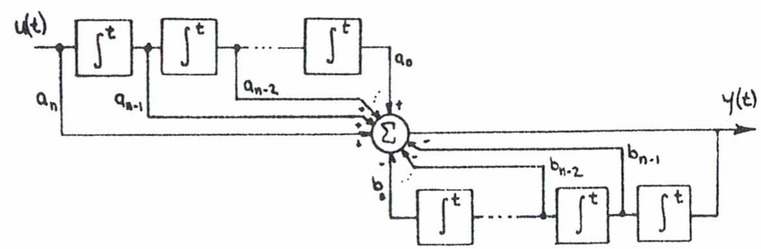
$$b_n y(t) + b_{n-1} \int^t y(t')dt' + \cdots + b_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\int^{t_2} y(t_1)\,dt_1\ldots dt$$

$$= a_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\int^{t_2} u(t_1)\,dt_1\ldots dt_n + \cdots + a_m \underbrace{\int^t\!\!\cdots\int^{t_2}}_{n-m \text{ fold}} u(t_1)\,dt_1\ldots dt$$

This system is sketched below, assuming that $n=m$ and with b_n normalized to 1 by dividing through the equation by b_n. If $m<n$, the coefficients a_k below with $k>m$ will be zero.

$$[1 + b_{n-1}D^{-1} + \cdots + b_1 D^{-n+1} + b_0 D^{-n}]\, y(t)$$

$$= [a_0 D^{-n} + a_1 D^{-n+1} + \cdots + a_n]\, u(t)$$



This system can now be solved using state variable methods. Note that the A matrix has dimension $2n \times 2n$.

(c) An equivalent n-integral system is shown in the block diagram below. Again, state variables may be used to solve for the output $y(t)$, here with only an $n \times n$ A-matrix. Using Laplace transforms, one can readily establish the equivalence of these two block diagrams.