Chapter 2

#2.1 (a) $e^{bk} = (e^b)^k$; $\quad L_A = s - e^b$

(b) $B\sinh ak = \frac{B}{2}(e^{ak} + e^{-ak})$; $\quad L_A = \underline{\underline{(s-e^a)(s-e^{-a})}}$
$= \underline{\underline{s^2 - 2\sinh a\, s + 1}}$

(c) $k^2 a^k + Ae^{bk}$; $\quad L_A = \underline{\underline{(s-a)^3(s-e^b)}}$

(d) $k a^k + A\sin bk$; $\quad L_A = \underline{\underline{(s-a)^2(s-e^{jb})(s-e^{-jb})}}$

#2.2 (a) $y_{k+2} + 7y_{k+1} + 12y_k = 0$

Char eqn: $\quad r^2 + 7r + 12 = 0$

i.e. $(r+4)(r+3) = 0$ $\quad \begin{cases} r_1 = -3 \\ r_2 = -4 \end{cases}$

$$\underline{\underline{y_k = c_1(-4)^k + c_2(-3)^k}}$$

(b) $y_{k+2} + 2y_{k+1} + 2y_k = 0$

Char eqn: $\quad r^2 + 2r + 2 = 0$ $\quad \begin{cases} r_1 = 1 + \sqrt{1-2} = 1+j \\ r_2 = 1-j = r_1^* \end{cases}$

$$y_k = \underline{\underline{c_1(1+j)^k + c_2(1-j)^k}}$$
$$= \underline{\underline{c_1\left(\sqrt{2}e^{j\pi/4}\right)^k + c_2\left(\sqrt{2}e^{-j\pi/4}\right)^k}}$$

(4 equivalent forms)

$$= \underline{\underline{c_1 2^{k/2} e^{j\pi k/4} + c_2 2^{k/2} e^{-j\pi k/4}}}$$
$$= \underline{\underline{\hat{c}_1 2^{k/2}\cos \pi k/4 + \hat{c}_2 2^{k/2} \sin \pi k/4}}$$

#2.2 (c) $\quad y_{k+2} + y_k = \sin k$

Since $\sin k = \dfrac{e^{jk} + e^{-jk}}{j2}$,

Choose $L_A = (s-e^j)(s-e^{-j})$

Then $(s-e^j)(s-e^{-j})(s^2+1) = 0$

The characteristic equation is

$$(r-e^j)(r-e^{-j})(r-e^{j\pi/2})(r-e^{-j\pi/2}) = 0$$

(where $e^{\pm j\pi/2} = \pm j$)

Thus

$$y_k = c_1 \cos\frac{\pi k}{2} + c_2 \sin\frac{\pi k}{2} + c_3\cos k + c_4 \sin k$$

The constants c_3 and c_4 are found from

$(s^2+1)(c_3\cos k + c_4 \sin k) \equiv$

i.e. $c_3[\cos(k+2) + \cos k] + c_4[\sin(k+2) + \sin k] \equiv$

$c_3 \cos k \cos 2 - c_3 \sin k \sin 2 + c_3 \cos k$
$\quad + c_4 \sin k \cos 2 - \cos k \sin 2 + c_4 \sin k \equiv$

i.e. $\cos k\,(c_3\cos 2 + c_3 - c_4 \sin 2)$
$\quad + \sin k\,(-c_3 \sin 2 + c_4\cos 2 + c_4) \equiv$

Thus $\quad c_3(1+\cos 2) - c_4 \sin 2 = 0$
$\quad c_3(-\sin 2) + c_4(1+\cos 2) = 1$

#2.2 (c) (cont)

$$c_3 = \frac{\begin{vmatrix} 0 & -\sin 2 \\ 1 & 1+\cos 2 \end{vmatrix}}{\begin{vmatrix} 1+\cos 2 & -\sin 2 \\ -\sin 2 & 1+\cos 2 \end{vmatrix}} = \frac{\sin 2}{1+2\cos 2+\cos^2 2-\sin^2 2}$$

$$= \frac{\sin 2}{2\cos 2(1+\cos 2)}$$

$$= \frac{\tan 2}{2(1+\cos 2)}$$

$$c_4 = \frac{\begin{vmatrix} 1+\cos 2 & 0 \\ -\sin 2 & 1 \end{vmatrix}}{2\cos 2(1+\cos 2)} = \frac{1}{2\cos 2}$$

soln:

$$y_k = c_1 \cos\frac{\pi k}{2} + c_2 \sin\frac{\pi k}{2} + \frac{\tan 2}{2(1+\cos 2)}\cos k + \frac{\sin k}{2\cos 2}$$

(d) $y_{k+2} - \frac{5}{2}y_{k+1} + y_k = 1$

use $L_A = S-1$ to annihilate the constant

Thus $(S^2 - \frac{5}{2}S+1)(S-1) = 0$

Char eqn $(r^2 - \frac{5}{2}r+1)(r-1) = 0$

$$(r-\tfrac{1}{2})(r-2)(r-1) = 0 \quad \begin{cases} r_1 = 1/2 \\ r_2 = 2 \\ r_3 = 1 \end{cases}$$

$$y_k = c_1(1/2)^k + c_2(2)^k + c_3$$

2.2 (d) (cont)

c_3 is found from $L[y_k] = 1$:

$$(S^2 - \tfrac{5}{2}S+1)\,c_3 = c_3(1-\tfrac{5}{2}+1) = c_3(-1/2) \equiv 1$$

$$\Rightarrow c_3 = -2$$

Thus $y_k = c_1(1/2)^k + c_2 2^k - 2$

with $y_0 = y_1 = 0$

$$\left.\begin{aligned} y_0 &= c_1 + c_2 - 2 \equiv 0 \\ y_1 &= \tfrac{1}{2}c_1 + 2c_2 - 2 \equiv 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 4/3 \\ c_2 &= 2/3 \end{aligned}$$

soln: $y_k = \frac{4}{3}(1/2)^k + \frac{2}{3}2^k - 2$

#2.3 $y_{k+2} - 2\tau y_{k+1} + y_k = 0$

Find solutions for above as τ varies. The auxiliary equation is: $r^2 - 2\tau r + 1 = 0$ with roots $r_1, r_2 = \tau \pm \sqrt{\tau^2-1}$.

(a) $\tau < -1$: The roots are distinct and real

$$y_k = c_1\left(\tau + \sqrt{\tau^2-1}\right)^k + c_2\left(\tau - \sqrt{\tau^2-1}\right)^k$$

(b) $\tau = -1$: The roots are repeated : $r_1 = r_2 = -1$

$$y_k = c_1(-1)^k + c_2 k(-1)^k$$

$$= (c_1 + c_2 k)(-1)^k$$

(c) $|\tau|<1$: The roots are distinct and complex

$$y_k = c_1\left(\tau+\sqrt{\tau^2-1}\right)^k + c_2\left(\tau-\sqrt{\tau^2-1}\right)^k$$

(d) $\tau=1$: The roots are repeated : $r_1 = r_2 = 1$

$$y_k = c_1 + c_2 k$$

(e) $\tau>1$: The roots are distinct and real

$$y_k = c_1\left(\tau+\sqrt{\tau^2-1}\right)^k + c_2\left(\tau-\sqrt{\tau^2-1}\right)^k$$

Perhaps a better representation for these cases can be obtained as follows:

(i) If $|\tau|>1$, the solution to the original difference equation is expressible as:

$$y_k = c_1 e^{\alpha k} + c_2 e^{-\alpha k}$$
$$= c_1' \cosh \alpha k + c_2' \sinh \alpha k$$

where $\tau = \begin{cases} \cosh\alpha & ,\tau>1 \\ -\cosh\alpha & ,\tau<-1 \end{cases}$

(ii) If $|\tau|<1$, then let $\tau=\cos\alpha$. The auxiliary equation is $r^2 - 2r\cos\alpha + 1 = 0$

with roots $r_1, r_2 = \cos\alpha \pm j\sin\alpha$

And so $y_k = c_1 \cos\alpha k + c_2 \sin\alpha k \quad , \tau=\cos\alpha$

(iii) For $\tau=1$ and $\tau=-1$ we obtain the solutions in (d) and (b), respectively (as before).

2.4 N_t = # messages possible in t-seconds

(a) $S_1=1$, $S_2=2$

$$N_t = N_{t-1} + N_{t-2} \quad \text{with } N_1 = 1 \ (S_1), \quad N_2 = 2 \ (S_1 S_1),\ (S_2)$$

ie $N_t - N_{t-1} - N_{t-2} = 0$

, Char eqn: $1 - r^{-1} - r^{-2} = 0$

ie $r^2 - r - 1 = 0$: $r_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.62$, $r_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.62$

Thus $N_t = c_1(1.62)^t + c_2(-.62)^t$

c_1 and c_2 are found from

$$N_1 = 1.62\,c_1 - .62\,c_2 = 1$$
$$N_2 = 2.61\,c_1 + .38\,c_2 = 2$$

$$c_1 = \frac{\begin{vmatrix} 1 & -.62 \\ 2 & .38 \end{vmatrix}}{\begin{vmatrix} 1.62 & -.62 \\ 2.61 & .38 \end{vmatrix}} = .72$$

$$c_2 = \frac{1}{.62}\left(1.62(.72) - 1\right) = .28$$

Thus $\lim_{t\to\infty}\left[\frac{\log_2 N_t}{t}\right] = \lim_{t\to\infty} \frac{\log_2\left[.72(1.62)^t + .28(-.62)^t\right]}{t}$

note that as $t\to\infty$, $(-.62)^t \to 0$

Thus $N_t \underset{t\to\infty}{\to} .72(1.62)^t$

$$\log_2\left[.72(1.62)^t\right] = \log_2(.72) + t\log_2(1.62)$$

hence

$$\lim_{t\to\infty}\left[\frac{\log_2 N_t}{t}\right] = \log_2 1.62$$

$$C = \frac{\log_{10} 1.62}{\log_{10} 2} = \frac{.208}{.303} = \underline{\underline{.69 \text{ bit/symb.}}}$$

(b) by inspection $N_t = 2^t$ $\quad (N_t = 2N_{t-1})$

hence $\underline{C = 1 \text{ bit/symbol}}$

#2.5 From the block diagram

(a) $y_k = u_k + 5y_{k-1} - 6y_{k-2}$

or $y_k - 5y_{k-1} + 6y_{k-2} = u_k \Rightarrow r^2 - 5r + 6 = 0$

$$(r-3)(r-2) = 0$$
$$r_1 = 3,\ r_2 = 2$$

The homogeneous solution is

$$y_k^{(h)} = c_1 3^k + c_2 2^k$$

Since $u_k = 2$, we try as a particular or steady-state solution $y_k^{(p)} = c_3$. Substituting we obtain

$$c_3 - 5c_3 + 6c_3 = 2 \Rightarrow 2c_3 = 2 \Rightarrow c_3 = 1$$

$\therefore$ steady-state output is 1 except that the so-called transient sol's ($c_1 3^k$ & $c_2 2^k$) do not die out with k. Thus for large k solution is $y_k = c_1 3^k + c_2 2^k + 1$. (Poorly designed Problem)

(2.5 cont.)

(b) $y_k - 2y_{k-1} + y_{k-2} = u_k \Rightarrow r^2 - 2r + 1 = 0$

$$(r-1)(r-1) = 0$$
$$r_1 = r_2 = 1$$

$$\therefore y_k^{(h)} = c_1 + c_2 k$$

For $u_k = 5 + 3k$ we see that this form satisfies the homogeneous equation. Thus multiply by k^2 and use the form

$$y_k^{(p)} = c_3 k^2 + c_4 k^3$$

Substitute $y_k^{(p)}$ into the original difference equation

$$(c_3 k^2 + c_4 k^3) - 2\left(c_3(k-1)^2 + c_4(k-1)^3\right) + c_3(k-2)^2 + c_4(k-2)^3 = 5 + 3k$$

Equating coefficients of like powers of k yields

$$c_3 = 1,\quad c_4 = 1/2$$

So that

$$y_k^{(p)} = k^2 + \tfrac{1}{2}k^3$$

The complete solution is $y_k = c_1 + c_2 k + k^2 + \frac{1}{2}k^3$ and since the roots of the auxiliary equation are 1,1 the "transient" solution again does not go to zero for large k.

(#2.6) The auxiliary equation is

$$r^2 + a_1 r + a_2 = 0$$

with roots

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

(a) real-valued roots: $a^2 > 4a_2$, ie., $a_2 < \frac{1}{4}a_1^2$

$$r_1 = -a_1 + \sqrt{a_1^2 - 4a_2} < 2$$

ie., $0 < \sqrt{a_1^2-4a_2} < 2+a_1$

(i) $-2 < a_1$

(ii) $\cancel{a_1^2} - 4a_2 < 4 + 4a_1 + \cancel{a_1^2}$

$$a_2 > -1 - a_1$$

$$r_2 = -a_1 - \sqrt{a_1^2-4a_2} > -2$$

ie., $2 - a_1 > \sqrt{a_1^2 - 4a_2} > 0$

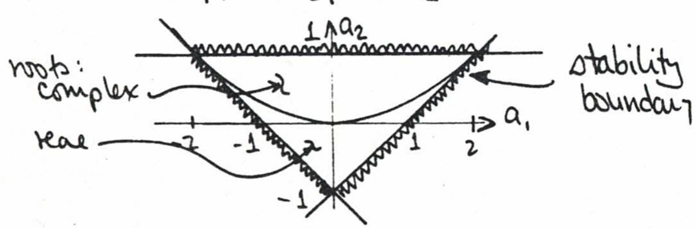
(i) $2 > a_1$

(ii) $4 - 4a_1 + \cancel{a_1^2} > \cancel{a_1^2} - 4a_2$

$$a_2 > -1 + a_1$$

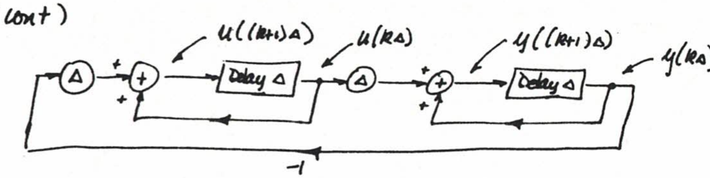
(b) complex-valued roots $a_2 > \frac{1}{4}a_1^2$

$$|r_1| = |r_2| = a_2 < 1$$



(#2.7) Replacing the integrators with the suggested approximate discrete-time system yields the following block diagram:

(2.7 cont)



Notation: Let $y(k\Delta) = y_k$ and $u(k\Delta) = u_k$. At the summing junctions we have:

$$\begin{cases} u_{k+1} = u_k - \Delta y_k \\ y_{k+1} = \Delta u_k + y_k \end{cases}$$

(with $u_0 = 0$ and $y_0 = 1$) the choice of these initial conditions is made so that $y_k = \cos k\theta$ as we shall see later.

Eliminate the u's from these equations.

$$y_{k+2} = \Delta u_{k+1} + y_{k+1} = \Delta(u_k - \Delta y_k) + y_{k+1}$$

$$= \Delta u_k - \Delta^2 y_k + y_{k+1}$$

$$= y_{k+1} - y_k - \Delta^2 y_k + y_{k+1}$$

$$\therefore \quad y_{k+2} - 2y_{k+1} + (1+\Delta^2)\, y_k = 0$$

The auxiliary equation is:

$$r^2 - 2r + (1+\Delta^2) = 0$$

with roots:

$$r_1, r_2 = \frac{2 \pm \sqrt{4 - 4(1+\delta^2)}}{2} = 1 \pm j\Delta$$

$$\rho = (1+\Delta^2)^{1/2}$$

$$\theta = \tan^{-1}\left(\frac{\Delta}{1}\right) \Rightarrow \Delta = \tan\theta, \quad (1+\Delta^2)^{1/2} = \frac{1}{\cos\theta}$$

(2.7 cont.)

$$\therefore\ y_k = c_1 \rho^k \cos k\theta + c_2 \rho^k \sin k\theta$$

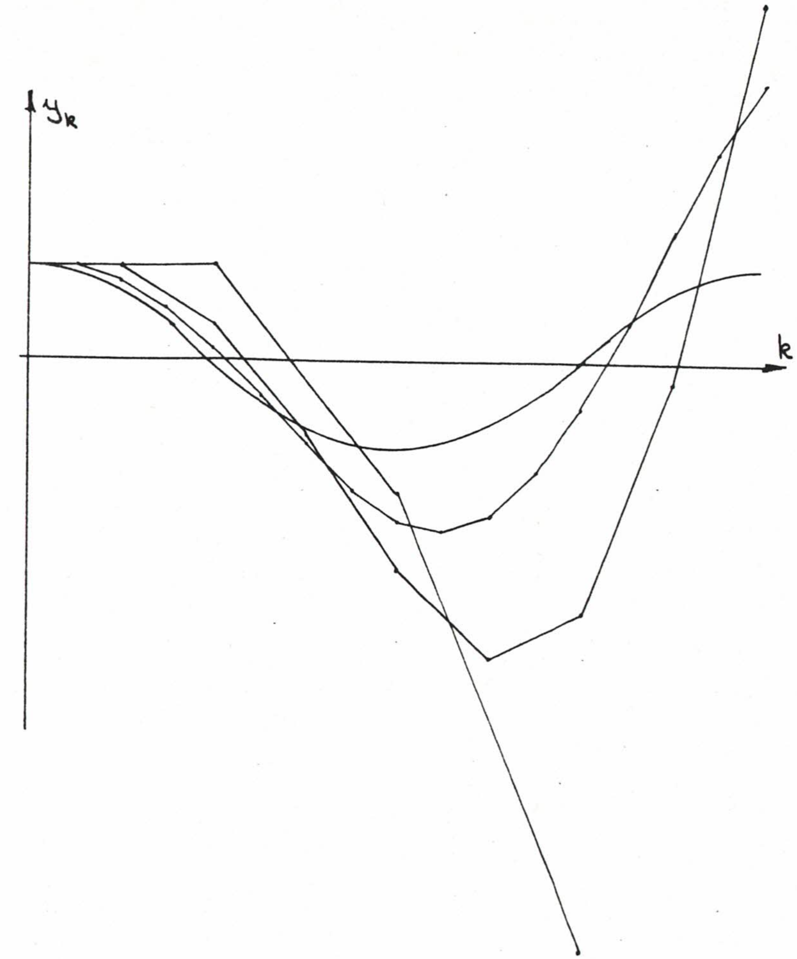
Since the continuous time solution we wish to approximate is cos t we would like to choose initial conditions so that $c_1 = 1$ and $c_2 = 0$. Thus choose $y_0 = 1$, $y_1 = 1$ ($\Rightarrow u_0 = 0$)

Thus $\quad y_0 = c_1 = 1$

and $\quad y_1 = \frac{1}{\cos\theta}\cos\theta + c_2 \frac{1}{\cos\theta}\sin\theta = 1 \ \Rightarrow c_2 = 0$

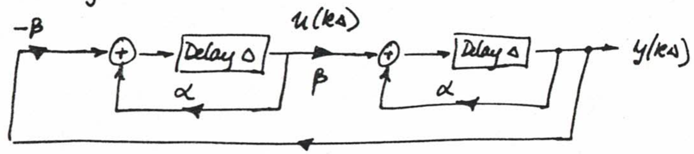
and $\quad y_k = \left(\frac{1}{\cos\theta}\right)^k \cos k\theta, \ k \geq 0.$ A plot of the approximations for various values of Δ is shown on the next page.

$\quad = y(k\Delta)$



(#2.8)

The Euler approximation of problem 2.7 provided a poor approximation for $\cos t$ in the last homework set. Consider the following:



For $\alpha=1$ and $\beta=\Delta$ the above is the Euler approximation of Problem 2.7. We wish to choose $\{\alpha, \beta, u_0, y_0\}$ so that $y_k = y(k\Delta) = \cos k\Delta$. At the summing junctions we have

$$\left.\begin{aligned} y_{k+1} &= \beta u_k + \alpha y_k \\ u_{k+1} &= \alpha u_k - \beta y_k \end{aligned}\right\} \text{①}$$

Eliminate the u's:

$$y_{k+1} = \beta u_k + \alpha y_k$$

$$y_{k+2} = \beta u_{k+1} + \alpha y_{k+1} = \beta(\alpha u_k - \beta y_k) + \alpha y_{k+1}$$

$$= \frac{\beta\alpha}{\beta}(y_{k+1} - \alpha y_k) - \beta^2 y_k + \alpha y_{k+1}$$

$$\therefore \quad y_{k+2} - 2\alpha y_{k+1} + (\alpha^2+\beta^2) y_k = 0 \quad \text{②}$$

But $y_k = \cos k\Delta$ is the solution to

$$y_{k+2} - 2(\cos\Delta)\, y_{k+1} + y_k = 0 \quad (\text{See Ex } 2.2) \quad \text{③}$$

(2.8 cont)

In other words, $(s^2 - 2\cos\Delta s + 1)$ annihilates $\{\cos k\Delta\}$. Thus set $\alpha = \cos\Delta$, $\beta = \sin\Delta$ so that ② and ③ correspond. Also

$$y_0 = \cos 0 = 1$$

$$y_1 = \cos\Delta = \beta u_0 + \alpha y_0 = \sin\Delta\, u_0 + \cos\Delta$$

$$\Rightarrow u_0 = 0.$$

Summary: $\alpha = \cos\Delta$, $\beta = \sin\Delta$, $y_0 = 1$, $u_0 = 0$.

(#2.9) From the block diagram:

$$y_k = \alpha u_k + \beta u_{k-1} + \gamma u_{k-2}$$

Let $u_k = e^{jk\omega T}$. Then $y_k = \alpha e^{jk\omega T} + \beta e^{j(k-1)\omega T} + \gamma e^{j(k-2)\omega T}$

$$y_k = e^{jk\omega T}(\alpha + \beta e^{-j\omega T} + \gamma e^{-2j\omega T})$$

$$\therefore \quad H(e^{j\omega T}) = \alpha + \beta e^{-j\omega T} + \gamma e^{-2j\omega T}$$

We want

$$H(e^{j\omega T})\Big|_{\omega T=0} = 1, \quad H(e^{j\omega T})\Big|_{\omega T=\frac{1}{10}} = 0$$

Thus

$$\begin{cases} \alpha+\beta+\gamma = 1 \\ \alpha + \beta e^{-j/10} + \gamma e^{-2j/10} = 0 \end{cases}$$

Suppose we choose $\alpha = \gamma$ (since we have more unknowns than equations). Then we can simplify our equations

(2.9 cont)

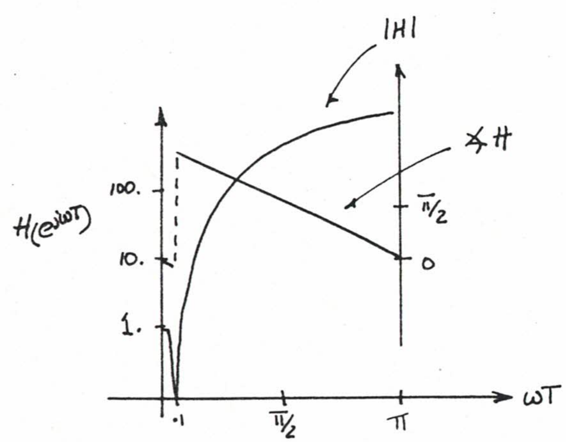
$$\begin{cases} 2\alpha + \beta = 1 \\ 2\alpha \cos(\tfrac{1}{10}) + \beta = 0 \end{cases}$$

Solving for α and β :
$$\alpha = \frac{1}{2(1-\cos(.1))}$$
$$\beta = \frac{-\cos(.1)}{1-\cos(.1)}$$

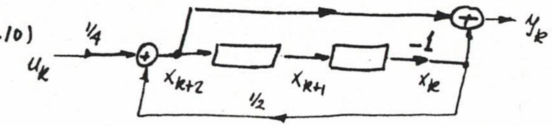
Thus
$$|H(e^{j\omega T})| = \left| \frac{\cos \omega T - \cos(.1)}{1-\cos(.1)} \right|$$

$$\measuredangle H(e^{j\omega T}) = \begin{cases} -\omega, & 0 \le \omega \le .1 \\ \pi - \omega, & .1 \le \omega \le \pi \end{cases}$$

<u>Rough Sketch</u>:



(2.10)



From the block diagram:
$$\left.\begin{aligned} y_k &= x_{k+2} - x_k \\ x_{k+2} &= -\tfrac{1}{2}x_k + \tfrac{1}{4}u_k \end{aligned}\right\} \text{ Eliminate the } x_k\text{'s.}$$

$$\left.\begin{aligned} y_k &= (S^2-1)\,x_k \\ (S^2+\tfrac{1}{2})\,x_k &= \tfrac{1}{4}u_k \end{aligned}\right\} \quad (S^2+\tfrac{1}{2})\,y_k = (S^2-1)\,\tfrac{1}{4}u_k$$

$$\Rightarrow \quad y_{k+2} + \tfrac{1}{2}y_k = \tfrac{1}{4}u_{k+2} - \tfrac{1}{4}u_k$$

And so,
$$H(e^{j\theta}) = \frac{-\frac{1}{4} + \frac{1}{4}e^{2j\theta}}{\frac{1}{2} + e^{2j\theta}} = \frac{\frac{1}{4} - \frac{1}{4}e^{-2j\theta}}{1 + \frac{1}{2}e^{-2j\theta}}$$

$$= \frac{1}{4}\left[\frac{-1+e^{2j\theta}}{\frac{1}{2}+e^{2j\theta}}\right]\left[\frac{\frac{1}{2}+e^{-2j\theta}}{\frac{1}{2}+e^{-2j\theta}}\right]$$

$$= \frac{1}{4}\left\{\frac{-\frac{1}{2} + \frac{1}{2}e^{2j\theta} - e^{-2j\theta} + 1}{(\frac{1}{2})^2 + \frac{1}{2}e^{-2j\theta} + \frac{1}{2}e^{2j\theta} + 1}\right\}$$

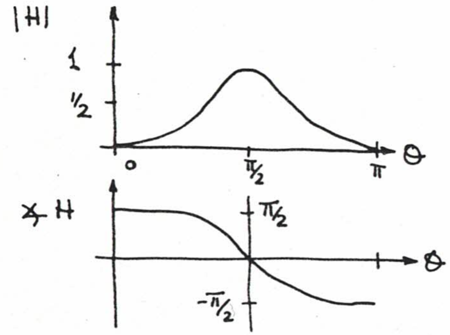
$$= \frac{1}{4}\left\{\frac{-\frac{1}{2} + \frac{1}{2}\cos 2\theta + \frac{1}{2}j\sin 2\theta - \cos 2\theta + j\sin 2\theta + 1}{\frac{5}{4} + \cos 2\theta}\right\}$$

(2.10 cont.)

$$|H|^2 = \frac{2 - 2\cos 2\theta}{16\,(5/4 + \cos 2\theta)}$$

$$\measuredangle H = \measuredangle(e^{j2\theta} - 1) - \measuredangle(e^{j2\theta} + 1/2)$$

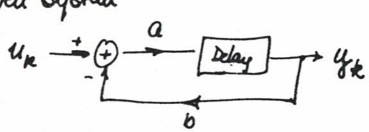
Rough Sketch:



#2.11 With $y_1(k) = \sum_{n=0}^{\infty} u(k-n)\, e^{-\alpha n}$

it must be that $h_1(k) = \begin{cases} (e^{-\alpha})^k & , k \geq 0 \\ 0 & , k < 0 \end{cases}$

For the given System



$$\frac{1}{a} y_{k+1} = u_k - b\, y_k$$

or $\quad y_{k+1} + ab\, y_k = a\, u_k$

(2.11 cont)

The impulse response is thus

$$h_2(k) = c\,(-ab)^k \quad \text{with } h_0 = 0,\ h_1 = a \quad \Rightarrow \quad c = -1/b$$

Thus $h_2(k) = \begin{cases} -\frac{1}{b}(-ab)^k, & k \geq 1 \\ 0 & , k < 1 \end{cases}$

Choosing $b = -e^{-\alpha}$ and $a = 1$ we obtain $h_2(k) = \begin{cases} e^{-\alpha(k-1)} & , k \geq 1 \\ 0 & ; k < 1 \end{cases}$

Note, however, this is a <u>delayed</u> version of the impulse response we want. This is a "close" as we can come.

#2.12

(a) $h_k = \begin{cases} (1/2)^k & , k \geq 0 \\ 0 & , k < 0 \end{cases}$

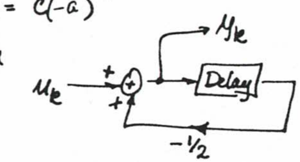
Since h_k is of the form a^k (with $a = 1/2$) we try a first order discrete-time system, i.e., try

$$y_k + a\, y_{k-1} = u_k$$

This difference equation has an impulse response

$$h_k = c(-a)^k$$

Thus use the system



(b) $\{h_k\} = \{1, 1, 1/2, 1/2, 1/4, 1/4, \ldots\}$

Notice the following: the above sequence can be obtained

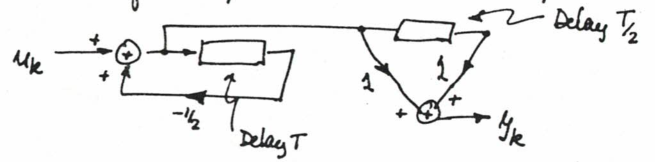
(2.12 cont.)

by convolving the sequences

$$\{1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \dots\} * \{1, 1, 0, 0, \dots\}$$

We can preform this convolution by convolving

$\{1, 1/2, 1/4, 1/8, \dots\}$ with $\{1, 1, 0, 0, \dots\}$ in which the "clock" of the sequence $\{1, 1, 0, 0, \dots\}$ runs twice the rate as the clock of the sequence $(1/2)^k$. Thus a system is:



*2.13

$$y_{k+2} - y_{k+1} + \tfrac{1}{4} y_k = u_k$$

$$r^2 - r + 1/4 = 0 \Rightarrow r_1, r_2 = 1/2 \Rightarrow h_k = c_1(1/2)^k + c_2 k (1/2)^k$$

Initial Conditions: $h_{k+2} - h_{k+1} + 1/4\, h_k = \delta_k$

$\therefore h_0 = 0,\ h_1 = 0,\ h_2 = 1,\ h_3 = \delta_1 + h_2 - 1/4\, h_1 = 1$

Using h_2 and h_3 (h_0 and h_1 are special cases) we have

$$\left.\begin{aligned} h_2 &= c_1(1/2)^2 + c_2(2)(1/2)^2 = \tfrac14 c_1 + \tfrac12 c_2 = 1 \\ h_3 &= c_1(1/2)^3 + c_2(3)(1/2)^3 = \tfrac18 c_1 + \tfrac38 c_2 = 1 \end{aligned}\right\} \begin{aligned} c_1 &= -4 \\ c_2 &= 4 \end{aligned}$$

$$\therefore \hat{u}_k = \begin{cases} -4(1/2)^k + 4k(1/2)^k, & k \ge 2 \\ 0, & k < 2 \end{cases} \leftarrow$$

(2.13 cont.)

check: $(S^2 - S + 1/4)\, h_k = \left[4(k+2-1)\tfrac14(1/2)^k\right]\xi_{k+1}$

$$- 4\left[(k+1-1)(1/2)(1/2)^k\right]\xi_k + \left[(k-1)(1/2)^k\right]\xi_{k-1}$$

$$= 0\cdot\delta_{k+1} + 1\cdot\delta_k + 0\cdot\delta_k + (k+1-2k+k-1)(1/2)^k \xi_{k-1}$$

$$= \delta_k \qquad \text{Recall: } \xi_{k-a} = \begin{cases} 1, & k \ge a \\ 0, & k < a \end{cases}.$$

(b) $(S^2 - \tfrac14)\, y_k = u_k$

From $r^2 - 1/4 = (r - 1/2)(r + 1/2) = 0$

$$r_1 = 1/2,\quad r_2 = -1/2$$

Thus

$$h_k = c_1(1/2)^k + c_2(-1/2)^k$$

The initial conditions are:

$$h_{k+2} - 1/4\, h_k = \delta_k \Rightarrow \begin{cases} h_0 = 0 \\ h_1 = 0 \\ h_2 = 1 \end{cases}$$

Thus

$$\left.\begin{aligned} c_1(1/2) + c_2(-1/2) &= 0 \\ c_1(1/4) + c_2(1/4) &= 1 \end{aligned}\right\} c_1 = c_2 = 2$$

And so,

$$h_k = \begin{cases} \left[1 + (-1)^k\right](1/2)^{k-1}, & k \ge 1 \\ 0, & k < 1 \end{cases}$$

(2.13 cont.)

check: $$(S^2 - 1/4)h_k = [1+(-1)^{k+2}](1/2)^{k+1}\xi_{k+1} - [1+(-1)^k](1/2)^{k-1}\xi_{k-1}$$
$$= 0\cdot\delta_{k+1} + 1\cdot\delta_k + 0 \quad, k\geq 1$$
$$= \delta_k.$$

(c) $y_k = u_k + 3y_{k-1} - 3y_{k-2} + y_{k-3}$

From $r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$

$\Rightarrow h_k = c_1 + c_2 k + c_3 k^2$

with initial conditions: $h_{-2} = 0,\ h_{-1} = 0,\ h_0 = 1$

Thus
$$\left.\begin{aligned} c_1 - 2c_2 + 4c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = 3/2,\ c_3 = 1/2$$
$c_1 = 1$

$$\Rightarrow h_k = 1 + \frac{3}{2}k + \frac{1}{2}k^2,\quad k\geq 0 \quad (\text{also for } -1,-2)$$

check: $$(1-3S^{-1}+3S^{-2}-S^{-3})h_k = (1+\tfrac{3}{2}k+\tfrac{1}{2}k^2)\xi_{k+2}$$
$$+ \left[-3 - \tfrac{9}{2}(k-1) - \tfrac{3}{2}(k-1)^2\right]\xi_{k+1}$$

(cont)

(2.13 cont)

$$+ \left[+3 + \tfrac{9}{2}(k-2) + \tfrac{3}{2}(k-2)^2\right]\xi_k$$
$$+ \left[-1 - \tfrac{3}{2}(k-3) - \tfrac{1}{2}(k-3)^2\right]\xi_{k-1}$$

Thus $(1-3S^{-1}+3S^{-2}-S^{-3})h_k =$

$$[0\cdot\delta_{k-2} + 0\cdot\delta_{k-1} + \delta_k + (1+\tfrac{3}{2}k+\tfrac{1}{2}k^2)]\xi_{k-1}$$
$$+ [0\cdot\delta_{k-1} + 0\cdot\delta_k + (-3 - \tfrac{9}{2}k + \tfrac{9}{2} - \tfrac{3}{2}k^2 + 3k - \tfrac{3}{2})]\xi_{k-1}$$
$$+ [0\cdot\delta_k + (3 + \tfrac{9}{2}k - 9 + \tfrac{3}{2}k^2 - 6k + 6)]\xi_{k-1}$$
$$+ [-1 - \tfrac{3}{2}k + \tfrac{9}{2} - \tfrac{1}{2}k^2 + 3k - \tfrac{9}{2}]\xi_{k-1}$$

$$= \delta_k + \Big\{(1 - 3 + \tfrac{9}{2} - \tfrac{3}{2} + 3 - 9 + 6 - 1 + \tfrac{9}{2} - \tfrac{9}{2})$$
$$+ k(\tfrac{3}{2} - \tfrac{9}{2} + 3 + \tfrac{9}{2} - 6 - \tfrac{3}{2} + 3)$$
$$+ k^2(\tfrac{1}{2} - \tfrac{3}{2} + \tfrac{3}{2} - \tfrac{1}{2})\Big\}.$$
$$= \delta_k$$

(d) $(1-3S^{-1}+3S^{-2}-S^{-3})y_k = S^{-3}u_k$

From (c) $h_k = L_0(\hat{h}_k) = S^{-3}(1+\tfrac{3}{2}k+\tfrac{1}{2}k^2) = (1-\tfrac{3}{2}k+\tfrac{1}{2}k^2)\cdot\xi_{k-3}$

#2.14 Answer given in text.

#2.15

$$\left(1-\frac{S^{-2}}{9}\right)[y_k] = (1/3)^k, \ k \geq 0$$

$$y_k - \frac{1}{9}y_{k-2} = (1/3)^k$$

(a) Direct Method:

$$r^2 - 1/9 = 0 \Rightarrow r^2 = 1/9 \Rightarrow r_1, r_2 = \pm 1/3$$

$$\therefore \ y_k^{(h)} = c_1(1/3)^k + c_2(-1/3)^k$$

And so $y_k^{(p)} = c_3 k (1/3)^k$. Thus

$$c_3 k(1/3)^k - \frac{1}{9}c_3(k-2)(1/3)^{k-2} = (1/3)^k$$

$$c_3 k(1/3)^k - \frac{1}{9}c_3 k\left(\frac{1}{3}\right)^k \cdot 9 + \frac{2}{9}c_3(1/3)^k \cdot 9 = (1/3)^k$$

$$\therefore \ 2c_3 = 1 \Rightarrow c_3 = 1/2$$

And so $y_k = c_1(1/3)^k + c_2(-1/3)^k + \frac{1}{2}k(1/3)^k$.

with $y_0 = y_1 = 0 \Rightarrow y_k = \begin{cases} -1/4(1/3)^k + 1/4(-1/3)^k + 1/2k(1/3)^k, & k \geq 0 \\ 0 & , k<0 \end{cases}$

(b) By Convolution:

Impulse response: $h_k = c_1(1/3)^k + c_2(-1/3)^k$

Initial conditions: $h_k - \frac{1}{9}h_{k-2} = \delta_k \Rightarrow \begin{array}{l} h_0 = 1 = c_1 + c_2 \\ h_1 = 0 = \frac{c_1}{3} - \frac{c_2}{3} \end{array}$

$$\Rightarrow c_1 = c_2 = 1/2$$

$$\Rightarrow h_k = \frac{1}{2}(1/3)^k + \frac{1}{2}(-1/3)^k.$$

(2.15 cont)

$$\therefore \ y_k = \sum_{n=0}^{k} h_n u_{k-n} = \frac{1}{2}\sum_{n=0}^{k}\left[\left(\frac{1}{3}\right)^n + \left(-\frac{1}{3}\right)^n\right]\cdot\left(\frac{1}{3}\right)^{k-n}$$

$$= \frac{1}{2}\sum_{n=0}^{k}\left[\left(\frac{1}{3}\right)^k + \left(\frac{1}{3}\right)^k(-1)^n\left(\frac{1}{3}\right)^n\left(\frac{1}{3}\right)^{-n}\right]$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^k \sum_{n=0}^{k}\left[1^n + (-1)^n\right] = \frac{1}{2}(1/3)^k k + \frac{1}{2}(1/3)^k \sum_{n=0}^{k}(-1)^n$$

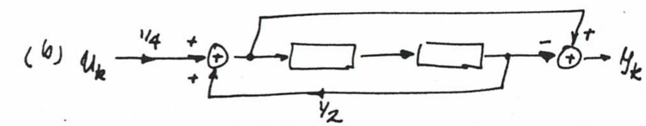
But $\sum_{n=0}^{k}(-1)^n = \begin{cases} -1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$. Thus the above can be written as

$$y_k = \begin{cases} -1/4(1/3)^k + 1/4(-1/3)^k + 1/2k(1/3)^k, & k \geq 0 \\ 0 & , k<0 \end{cases}$$

#2.16

(a) $\{h_k\} = \{\alpha, \beta, \gamma\}$ from the block diagram.

(b)



From Problem 2.10, $y_{k+2} + \frac{1}{2}y_k = \frac{1}{4}u_{k+2} - \frac{1}{4}u_k$

Thus $r^2 + 1/2 = 0 \Rightarrow r_1, r_2 = \pm j\frac{1}{\sqrt{2}}$

$$\therefore \ h_k = c_1\left(\frac{1}{\sqrt{2}}\right)^k \cos\frac{k\pi}{2} + c_2\left(\frac{1}{\sqrt{2}}\right)^k \sin\frac{k\pi}{2}$$

Initial Conditions: $h_{k+2} + \frac{1}{2}h_k = \frac{1}{4}\delta_{k+2} - \frac{1}{4}\delta_k$

$$h_0 + \frac{1}{2}h_{-2} = \frac{1}{4}\delta_0 - \frac{1}{4}\delta_{-2} \Rightarrow h_0 = 1/4$$

(2.16 cont)

Similarly, $h_1 = 0$, $h_2 = -3/8$

$$h_1 = 0 = \frac{1}{\sqrt{2}}(C_1 \cdot 0 + C_2) \Rightarrow C_2 = 0$$

$$h_2 = -\frac{3}{8} = \left(\frac{1}{\sqrt{2}}\right)^2 \left[C_1(-1) + 0\right] \Rightarrow C_1 = 3/4$$

And so,
$$h_k = \begin{cases} 0 & k<0 \\ 1/4 & k=0 \\ \frac{3}{4}\left(\frac{1}{\sqrt{2}}\right)^k \cos\frac{k\pi}{2} & , k>0 \end{cases}$$

#2.17
$$y_k - 4y_{k-1} + 3y_{k-2} = -4u_k + u_{k-1}$$

$$r^2 - 4r + 3 = 0 \Rightarrow r_1 = 1, \; r_2 = 3$$

$$\Rightarrow h_k = c_1 + c_2 3^k$$

<u>Initial Conditions</u>: $h_k - 4h_{k-1} + 3h_{k-2} = -4\delta_k + \delta_{k-1}$

$$\Rightarrow h_0 = -4, \; h_1 = -15$$

$$\Rightarrow c_1 = 3/2, \; c_2 = -11/2$$

$$\therefore \; h_k = \begin{cases} \frac{3}{2} - \frac{11}{2} 3^k & , k \ge 0 \\ 0 & , k<0 \end{cases}$$

#2.18 there are three systems in cascade each with impulse responsefunction: $h_k = \begin{cases} 1 & , k \ge 1 \\ 0 & ; k<1 \end{cases}$

(#2.18 cont)

Thus $h_A(k) = \begin{cases} 1, & k \ge 1 \\ 0, & k<1 \end{cases}$

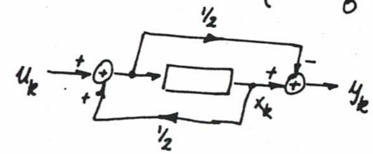
And so, $h_B(k) = h_A(k) * h(k)$

$$= \begin{cases} \sum_{m=1}^{k-1} 1 \cdot 1 = k-2 & , k \ge 2 \\ 0 & , k<2 \end{cases}$$

Continuing, $h_C(k) = h_B(k) * h(k)$

$$= \begin{cases} \sum_{m=2}^{k-2} (m-2) = \frac{(k-1)(k-2)}{2} & , k \ge 3 \\ 0 & , k<3 \end{cases}$$

#2.19



At the summing nodes:

$$\begin{cases} y_k = -\frac{1}{2}x_{k+1} + x_k \\ x_{k+1} = u_k + \frac{1}{2}x_k \end{cases} \quad \text{in operator notation:} \quad \begin{cases} y_k = (-\frac{S}{2}+1)[x_k] \\ u_k = (S-\frac{1}{2})[x_k] \end{cases}$$

And so $(S-\frac{1}{2})[y_k] = (-\frac{S}{2}+1)(S-\frac{1}{2})[x_k] = (-\frac{S}{2}+1)[u_k]$

Thus $y_{k+1} - \frac{1}{2}y_k = -\frac{1}{2}u_{k+1} + u_k$ or $y_k - \frac{1}{2}y_{k-1} = -\frac{1}{2}u_k + u_{k-1}$

(b) $r - \frac{1}{2} = 0 \Rightarrow r = \frac{1}{2} \Rightarrow h_k = C(\frac{1}{2})^k$ with initial conditions $h_0 = -\frac{1}{2}$, $h_1 = \frac{3}{4} = C_1(\frac{1}{2}) \Rightarrow C_1 = 3/2$ $\;\therefore\; h_k = \begin{cases} -1/2 & , k=0 \\ 3/2(1/2)^k & ; k>0 \\ 0 & ; k<0 \end{cases}$

(2.19 cont)

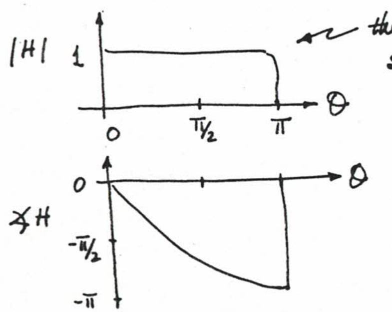
$$H(e^{j\theta}) = \frac{-\frac{1}{2} + e^{-j\theta}}{1 - \frac{1}{2}e^{-j\theta}} = \frac{-\frac{1}{2}e^{j\theta} + 1}{e^{j\theta} - \frac{1}{2}}, \quad \theta = \omega T$$

Thus
$$|H(e^{j\theta})| = \frac{|-\frac{1}{2}e^{j\theta} + 1|}{|e^{j\theta} - \frac{1}{2}|} = \frac{\left[(-\frac{1}{2}\cos\theta + 1)^2 + (-\frac{1}{2}\sin\theta)^2\right]^{1/2}}{\left[(\cos\theta - \frac{1}{2})^2 + \sin^2\theta\right]^{1/2}}$$

$$= \frac{\left[\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta\right]^{1/2}}{\left[\cos^2\theta - \cos\theta + \frac{1}{4} + \sin^2\theta\right]^{1/2}} = \frac{(\frac{5}{4} - \cos\theta)^{1/2}}{(\frac{5}{4} - \cos\theta)^{1/2}} = 1, \quad \forall\theta$$

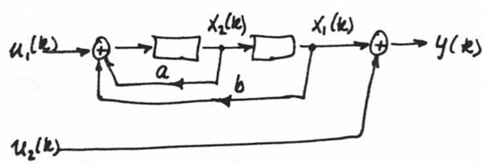
And
$$\measuredangle H(e^{j\theta}) = \measuredangle(-\tfrac{1}{2}e^{j\theta} + 1) - \measuredangle(e^{j\theta} - \tfrac{1}{2})$$

Sketch:



this is called an "all pass" system.

\# 2.20



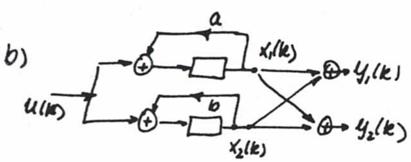
(2.20 cont)

From the block diagram we have:

$$\left.\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= b x_1(k) + a x_2(k) + u_1(k) \\ y(k) &= x_1(k) + u_2(k) \end{aligned}\right\} \quad \underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{u}$$

$$y(k) = [1 \ 0]\,\underline{x}(k) + [0 \ 1]\,\underline{u}(k)$$

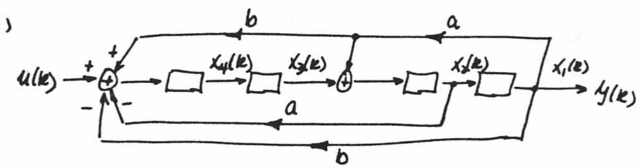
(b)



$$\begin{cases} x_1(k+1) = a x_1(k) + u(k) \\ x_2(k+1) = b x_2(k) + u(k) \\ y_1(k) = x_1(k) + x_2(k) \\ y_2(k) = x_1(k) + x_2(k) \end{cases}$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$
$$\underline{y}(k) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x}(k)$$

(c)

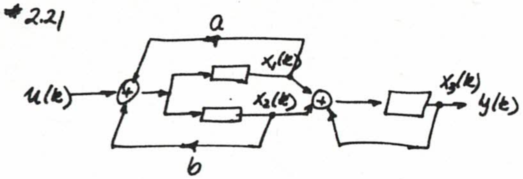


$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= a x_1(k) + x_3(k) \\ x_3(k+1) &= x_4(k) \\ x_4(k+1) &= -b x_1(k) - a x_2(k) + a b x_1(k) + u(k) \\ y(k) &= x_1(k) \end{aligned}$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ ab-b & -a & 0 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \ 0 \ 0 \ 0]\ \underline{x}(k)$$

#2.21



$x_1(k+1) = a\,x_1(k) + b\,x_2(k) + u(k)$
$x_2(k+1) = a\,x_1(k) + b\,x_2(k) + u(k)$
$x_3(k+1) = x_1(k) + x_2(k) + x_3(k)$

(a) $y(k) = x_3(k)$

$$\underline{x}(k+1) = \begin{bmatrix} a & b & 0 \\ a & b & 0 \\ 1 & 1 & 1 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [0 \ \ 0 \ \ 1]\, \underline{x}(k)$$

(b) $\det A = \det \begin{vmatrix} a & b & 0 \\ a & b & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$ (2 rows identical)

(c) $g(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b & 0 \\ a & b-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)(\lambda)(\lambda - a - b) = 0$

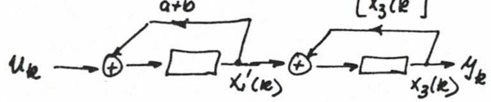
Eigenvalues: $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = a+b$

(d) Add states x_1 and x_2 to form x_1':

$$\begin{bmatrix} x_1' \\ x_3 \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = (0 \ \ 1] \begin{bmatrix} x_1'(k) \\ x_3(k) \end{bmatrix}$$

Note: If $\det A = 0$, then the rows of A are not linearly independent. Thus one state variable can be written in terms of the others. The degree of the system is $n-1$ (or less).



#2.22

Given $f(\lambda) = \sum_m \alpha_m \lambda^m$.

(a) $\underline{A} = \sum_{i=1}^{n} \lambda_i \underline{E}_i$ & $\underline{f}(\underline{A}) = \sum_{i=1}^{n} f(\lambda_i)\, \underline{E}_i$

then $\underline{A}\,\underline{f}(\underline{A}) = \underline{A} \sum_{i=1}^{n} f(\lambda_i)\underline{E}_i = \sum_{i=1}^{n} f(\lambda_i)\, \underline{A}\,\underline{E}_i = \sum_{i=1}^{n} f(\lambda_i)\underline{E}_i \underline{A} = \underline{f}(\underline{A})\,\underline{A}$

(similarly for repeated roots since $\underline{N}_i$ and $\underline{A}$ commute also.)

(b) $g(t) = \frac{d}{dt} f(t)$

Prove: $\frac{d}{dt} f(At) = A\, g(At) = A \frac{d}{dt} f(At) = A f'(At)$

$f(\lambda) = \sum_m \alpha_m \lambda^m$

$f(At) = \sum_m \alpha_m (At)^m = \sum_m \alpha_m t^m A^m$

$\frac{d}{dt} f(At) = \sum_m m\, \alpha_m t^{m-1} A^m$

$= A \sum_m \alpha_m\, m\, (tA)^{m-1} = A f'(At)$

since $f'(\lambda) = \sum_m \alpha_m\, m\, \lambda^{m-1}$.

#2.23 Suppose $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $g(\lambda) = 0$ has real roots $\alpha \pm \beta$

with $\lambda_1 = \alpha + \beta$, $\lambda_2 = \alpha - \beta$. Then $A = \lambda_1 E_1 + \lambda_2 E_2$ with

$$E_1 = \frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda_2 I}{\lambda_1 - \lambda_2}, \quad E_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$$

(2.23 Cont)

Thus $E_1 = \begin{bmatrix} \frac{a_{11}-\alpha+\beta}{2\beta} & \frac{a_{12}}{2\beta} \\ \frac{a_{21}}{2\beta} & \frac{a_{22}-\alpha+\beta}{2\beta} \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{a_{11}-\alpha-\beta}{-2\beta} & \frac{a_{12}}{-2\beta} \\ \frac{a_{21}}{-2\beta} & \frac{a_{22}-\alpha-\beta}{-2\beta} \end{bmatrix}$

Thus $A^k = (\alpha+\beta)^k E_1 + (\alpha-\beta)^k E_2$ (in general)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 3/4-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = 3/4,\ \lambda_2 = 1/2$

$A = 3/4\, E_1 + 1/2\, E_2 \qquad E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \qquad E_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

$\therefore A^k = (3/4)^k \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + (1/2)^k \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1/16 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 1/4 \\ \frac{1}{16} & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{1}{\sqrt{48}}$, $\lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{48}}$

Let $\alpha = 1/2$, $\beta = \frac{1}{\sqrt{48}}$

Using general result at top of page we have:

$A^k = \left(\frac{1}{2} + \frac{1}{\sqrt{48}}\right)^k \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{4} & 1 \end{bmatrix} + \left(\frac{1}{2} - \frac{1}{\sqrt{48}}\right)^k \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ -\frac{\sqrt{3}}{4} & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{\sqrt{2}}{2}$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{2}}{2}$

$\Rightarrow \alpha = 1/2 \quad \beta = \frac{\sqrt{2}}{2}$

then $A^k = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1/2 \end{bmatrix} + \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{-1}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1/2 \end{bmatrix}$

(2.23 cont)

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det \begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \lambda_2 = 1/2$

$A = \lambda_1 E_1 + N_1 \quad ; \quad A^k = \lambda_1^k E_1 + k(\lambda_1)^{k-1} N_1$

$A^0 = I = E_1 \quad ; \quad A = \frac{1}{2} I + N_1, \quad N_1 = A - 1/2\, I = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$

$\therefore A^k = \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\left(\frac{1}{2}\right)^{k-1} \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}.$

(e) $A = \begin{bmatrix} 3/4 & -1/2 \\ -15/32 & 1/2 \end{bmatrix}$, $g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 9/8,\ \lambda_2 = 1/8$

$E_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = 1/2 \begin{bmatrix} 5/4 & -1 \\ \frac{15}{16} & 3/4 \end{bmatrix}$, $E_2 = \frac{1}{2} \begin{bmatrix} 3/8 & 1 \\ \frac{15}{16} & 5/4 \end{bmatrix}$

$\therefore A^k = \left(\frac{9}{8}\right)^k E_1 + \left(\frac{1}{8}\right)^k E_2.$

*2.24

$x(n+1) = 3x(n) + 5y(n) + 2z(n)$
$y(n+1) = x(n) - y(n) + z(n)$
$z(n+1) = 2x(n) + y(n) + 3z(n)$

then $\underline{v}(n+1) = \begin{bmatrix} 3 & 5 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \underline{v}(n)$ with $\underline{v}(n) = \begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix}$

Our solution is thus $\underline{v}(n) = \underline{A}^n \underline{v}(0)$. We need A^n.

$g(\lambda) = \det[A - \lambda I] = \lambda^3 - 5\lambda^2 - 7\lambda + 11 = 0 \Rightarrow \lambda_1 = 1$, $\lambda_2 = 2 + \sqrt{15}$, $\lambda_3 = 2 - \sqrt{15}$

(2.24 cont)

Using the formulation based on the Caley-Hamilton theorem we apply

$$\left.\begin{aligned} \lambda_1^n &= \beta_0 + \beta_1\lambda_1 + \beta_2\lambda_1^2 \\ \lambda_2^n &= \beta_0 + \beta_1\lambda_2 + \beta_2\lambda_2^2 \\ \lambda_3^n &= \beta_0 + \beta_1\lambda_3 + \beta_2\lambda_3^2 \end{aligned}\right\} \Rightarrow \begin{aligned} \beta_0 &= .786 - .5(\lambda_1)^n + .264(\lambda_3)^n \\ \beta_1 &= .286 + .023(\lambda_1)^n - .309(\lambda_3)^n \\ \beta_2 &= -.071 + .026(\lambda_2)^n + .045(\lambda_3)^n \end{aligned}$$

And so $\underline{U}(n) = [\beta_0 I + \beta_1 A + \beta_2 A^2]\begin{bmatrix}1\\0\\0\end{bmatrix}$, $A^2 = \begin{bmatrix} 18 & 12 & 17 \\ 4 & 7 & 4 \\ 13 & 12 & 14 \end{bmatrix}$

so that $\underline{U}(n) = \begin{bmatrix} \beta_0 + 3\beta_1 + 18\beta_2 \\ \beta_1 + 4\beta_2 \\ 2\beta_1 + 13\beta_2 \end{bmatrix}$

$$\simeq \begin{bmatrix} .357 + .499(5.873)^n + .146(-1.873)^n \\ .129(5.873)^n - .129(-1.873)^n \\ -.357 + .391(5.873)^n - .034(-1.873)^n \end{bmatrix}, \ n \geq 0$$

#2.25 (see after 2.26)

#2.26

(a) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $g(\lambda) = \det[A - \lambda I] = 0$

$\Rightarrow \lambda_1, \lambda_2 = \cos\theta \pm j\sin\theta = e^{\pm j\theta}$

$\lambda_1 = e^{j\theta}$, $\lambda_2 = e^{-j\theta}$

$A^k = \lambda_1^k E_1 + \lambda_2^k E_2$, $A^k = e^{jk\theta} E_1 + e^{-jk\theta} E_2$

with $E_1 = \dfrac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \begin{bmatrix} 1/2 & -1/2j \\ 1/2j & 1/2 \end{bmatrix}$; $E_2 = \dfrac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \begin{bmatrix} 1/2 & +1/2j \\ -1/2j & 1/2 \end{bmatrix}$

$\therefore A^k = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$

Note: A is rotation matrix of 2 vector. A^k is a "k rotation".

(2.26 cont)

(b) $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\lambda_1 = re^{j\theta} = \alpha + j\beta$, $\lambda_2 = re^{-j\theta} = \alpha - j\beta$

E_1 and E_2 are same as in (a).

$A^k = r^k \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$

(c) $A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \alpha I \Rightarrow A^k = \alpha^k I$

(d) $A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = \alpha$ & so $A = \alpha E + N$

and $A^k = \alpha^k E + k\alpha^{k-1} N$

where $I = E$ and $A = \alpha I + N \Rightarrow N = A - \alpha I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Thus $A^k = \alpha^k I + k\alpha^{k-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha^k & k\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix}$.

(e) $A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha+\beta \end{bmatrix} \Rightarrow \lambda_1 = \alpha, \lambda_2 = \alpha + \beta$

$E_1 = \begin{bmatrix} 1 & -1/\beta \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1/\beta \\ 0 & 1 \end{bmatrix}$

And so $A^k = \alpha^k E_1 + (\alpha+\beta)^k E_2 = \begin{bmatrix} \alpha^k & \dfrac{(\alpha+\beta)^k - \alpha^k}{\beta} \\ 0 & (\alpha+\beta)^k \end{bmatrix}$

#2.25 Let $\underline{U}(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$. Then $\underline{U}(n+1) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \underline{U}(n)$

And so $\underline{U}(n) = A^n \underline{U}(0)$. The eigenvalues of A are:

$g(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda_1 = 4, \lambda_2 = -1$.

Then

$$\left.\begin{aligned} 4^n &= \beta_0 + 4\beta_1 \\ (-1)^n &= \beta_0 - \beta_1 \end{aligned}\right\} \Rightarrow \beta_0 = \frac{4(-1)^n + 4^n}{5}, \ \beta_1 = \frac{4^n - (-1)^n}{5}$$

Thus $A^n = \beta_0 I + \beta_1 A$ and $\underline{v}(n) = A^n \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\beta_0+\beta_1\\3\beta_1\end{bmatrix}$

and hence $x(n) = \frac{1}{5}\left[2\cdot 4^n + 3(-1)^n\right]$

$$y(n) = \frac{3}{5}\left[4^n - (-1)^n\right].$$

*2.27 Defining $x_1(k)$ as the state variable for the upper delay and $x_2(k)$ as the state variable for the lower delay, we have

$$A = \begin{bmatrix}1/2 & 0\\ 1/2 & 1/3\end{bmatrix},\quad B = \begin{bmatrix}1\\1\end{bmatrix},\quad C = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad D = 0$$

Now $h_k = \begin{cases} D & k=0 \\ CA^{k-1}B, & k>0\end{cases}$

To find A^k: $g(\lambda) = \det[A - \lambda I] = (1/2-\lambda)(1/3-\lambda) = 0$

$\Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 1/3 \quad \Rightarrow E_1 = \dfrac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$

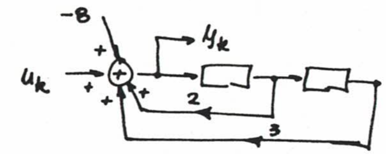
and $E_2 = \begin{bmatrix}0 & 0\\ -3 & 1\end{bmatrix}$

$$\therefore\ A^k = (1/2)^k \begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix} + (1/3)^k \begin{bmatrix}0 & 0\\ -3 & 1\end{bmatrix}$$

Hence $CA^{k-1}B = (\tfrac{1}{2})^{k-1} C\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}B + (1/3)^{k-1} C\begin{bmatrix}0 & 0\\ -3 & 1\end{bmatrix}B$

And so $h_k = \begin{cases} 0, & k=0 \\ \begin{bmatrix}(1/2)^k \\ 3(1/2)^k - 2(1/3)^k\end{bmatrix}, & k>0\end{cases}$

2.28

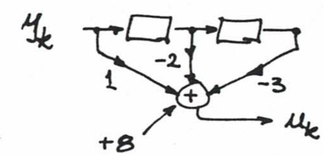


Difference equation: $y_k = u_k + 2y_{k-1} + 3y_{k-2} - 8$

To find inverse system solve for u_k as output. Thus

$$u_k = y_k - 2y_{k-1} - 3y_{k-2} + 8$$

A block diagram of this difference equation with y_k as input and u_k as the output is



← the inverse system

Note: If $H(e^{j\theta})$ is transfer function of original system and $G(e^{j\theta})$ is the transfer function of the inverse system, then it must be that $H(e^{j\theta})\,G(e^{j\theta}) = 1$. Now

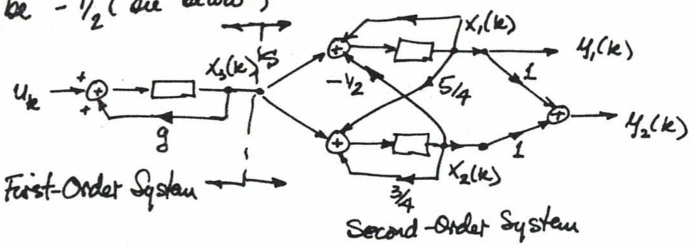
$$H(e^{j\theta}) = \frac{1}{1-2e^{-j\theta}-3e^{-j\theta 2}} \ \ \& \ \ G(e^{j\theta}) = 1 - 2e^{-j\theta} - 3e^{-2j\theta}$$

(We interpret -8 & +8 as initial conditions which we set to zero.)

*2.29 There is an error (typo) in this problem which if not corrected makes the overall system unstable and complicates the arithmetic. In the cross multiplier

(\# 2.29 cont)

in the second order system the multiplier of ½ should be $-\frac{1}{2}$ (see below)



(a)

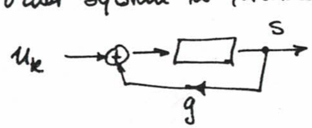
Entire System:

$$A = \begin{bmatrix} -1 & -1/2 & 1 \\ 5/4 & 3/4 & 1 \\ 0 & 0 & g \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = 0$$

Second-Order System:

$$A = \begin{bmatrix} -1 & -1/2 \\ 5/4 & 3/4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D = 0$$

(b) The gain g does not affect the stability of the second order system. The enti system is stable provided the first order system is stable.



$$y_{k+1} - g\,y_k = u_k$$

$$\Rightarrow r - g = 0 \Rightarrow r = g$$

$$\therefore \text{ stable iff } |g| < 1$$

(c) For $\{u_k\} = \{1, -a\}$, the sequence at point S is

$$y_s(k) = \{1, -a\} * h_s(k)$$

(\# 2.29 cont)

where $h_s(k)$ is the impulse response of the first order system and is given by

$$h_s(k) = \begin{cases} 0, & k \le 0 \\ g^{k-1}, & k > 0 \end{cases}$$

Thus

$$y_s(k) = \{1, -a\} * \{0, 1, g, g^2, \cdots\} = \begin{cases} 0, & k = 0 \\ 1, & k = 1 \\ g^{k-1}\left(1 - \frac{a}{g}\right), & k \ge 2 \end{cases}$$

(d) To find output sequences we can convolve $y_s(k)$, the output of the first order system, with the impulse response of the second-order system.

$$A = \begin{bmatrix} -1 & -1/2 \\ 5/4 & 3/4 \end{bmatrix} \Rightarrow g(\lambda) = \det(A - \lambda I) = \lambda^2 + \tfrac{1}{4}\lambda - \tfrac{1}{8} = 0$$

$$\Rightarrow \lambda_1 = -\tfrac{1}{2} \quad \lambda_2 = \tfrac{1}{4}$$

$$A = \lambda_1 E_1 + \lambda_2 E_2 \quad \text{where} \quad E_1 = \tfrac{1}{3}\begin{pmatrix} 5 & 2 \\ -5 & -2 \end{pmatrix}, \quad E_2 = \tfrac{1}{3}\begin{pmatrix} -2 & -2 \\ 5 & 5 \end{pmatrix}$$

$$\therefore A^k = \left(-\tfrac{1}{2}\right)^k E_1 + \left(\tfrac{1}{4}\right)^k E_2$$

And so

$$A^{k-1}B = A^{k-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7\left(-\frac{1}{2}\right)^{k-1} + (-4)\left(\frac{1}{4}\right)^{k-1} \\ -7\left(-\frac{1}{2}\right)^{k-1} + 10\left(\frac{1}{4}\right)^{k-1} \end{bmatrix}$$

Now $A^{k-1}B$ is the impulse response from the input (point S) to the state registers $x_1(k)$ and $x_2(k)$. Thus to obtain $y_1(k)$ we convolve $y_s(k)$ and $\frac{1}{3}\left[7\left(-\frac{1}{2}\right)^{k-1} - 4\left(\frac{1}{4}\right)^{k-1}\right]$. To obtain $y_2(k)$ we convolve $y_s(k)$ with the sum of the two components in $A^{k-1}B$ above. Namely, $2\left(\frac{1}{4}\right)^{k-1}$.

(# 2.29 cont)

Thus $y_2(k) = \{0, 1, \dots, g^{k-1}(1-\frac{a}{g}), \dots\} * \{0, 0, \dots, 2(\frac{1}{4})^{k-1}, \dots\}$

$= \{0, 0, 0, \dots, 2(\frac{1}{4})^{k-1} * g^{k-1}(1-a/g), \dots\}$

general term:

$$y_2(k) = \sum_{n=2}^{k-2} (1-\frac{a}{g}) \cdot \frac{g^n}{g} \cdot 8(\frac{1}{4})^{k-n}, \quad k \geq 2$$

$$= (1-\frac{a}{g})\frac{8}{g} \sum_{n=2}^{k-2} (4g)^n (\frac{1}{4})^k, \quad k \geq 2$$

with a good deal of algebra ?

$$= \begin{cases} \frac{g-a}{g^2} \cdot \frac{8}{1-4g}\left[16g^2(\frac{1}{4})^k - \frac{g^k}{4g}\right], & k \geq 2 \\ 0, & k < 2 \end{cases}$$

For $y_1(k)$ we have:

$$y_1(k) = \{0, 0, \dots, g^{k-1}(1-\frac{a}{g}), \dots\} * \left\{\frac{7}{3}(-\frac{1}{2})^{k-1} - \frac{4}{3}(\frac{1}{4})^{k-1}\right\}_{k \geq 2}$$

$$= \sum_{n=2}^{k-2} \frac{g-a}{g^2} g^n \left[\frac{7}{3}(-\frac{1}{2})^{k-n}(-2) - \frac{16}{3}(\frac{1}{4})^k 4^n\right]$$

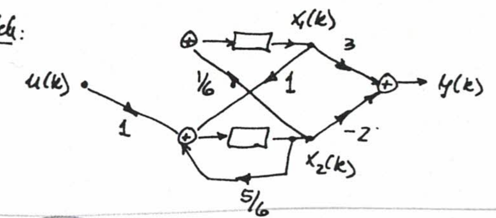
$$= \frac{g-a}{g^2}(\frac{-14}{3})(-\frac{1}{2})^k \cdot \sum_{n=2}^{k-2}(-2g)^n + (\frac{g-a}{g^2})(-\frac{16}{3})(\frac{1}{4})^k \cdot \sum_{n=2}^{k-2}(4g)^n$$

$$= (\frac{g-a}{g^2})(\frac{-14}{3})(-\frac{1}{2})^k \cdot \left[\frac{8g^3 + (-2g)^k}{2g(1+2g)}\right] + (\frac{g-a}{g^2})(-\frac{16}{3})(\frac{1}{4})^k \cdot \left[\frac{64g^3 - (4g)^k}{4g(1-4g)}\right]$$

(# 2.30)

$A = \begin{bmatrix} 0 & 1 \\ 1/6 & 5/6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [3 \ \ -2], \quad D = 0$

Sketch:



With $x' = T^{-1}x$ & $T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

incorrect assumption

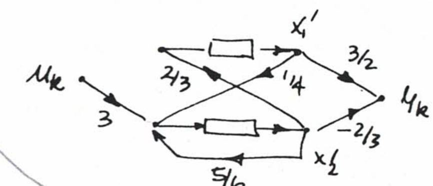
and $T: \quad A \to T^{-1}AT = \begin{bmatrix} 0 & 2/3 \\ 1/4 & 5/6 \end{bmatrix}$

$B \to T^{-1}B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$C \to CT = [3/2 \ \ -2/3]$

$D \to D = 0$

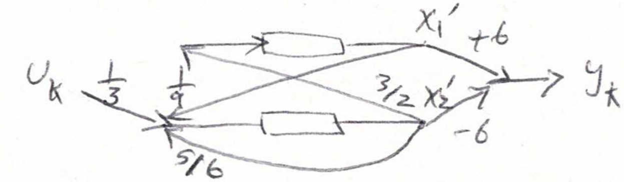
Notice we obtain the same topological structure – the diagonal transformation T^{-1} has served to change only the component values of the multipliers. This diagonal transformation can be used to scale fixed point arithmetic, digital structures.



Correct the above solution:

$x' = TX \Rightarrow A' = TAT^{-1} = \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{1}{9} & \frac{5}{6} \end{bmatrix}, B' = TB = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}$

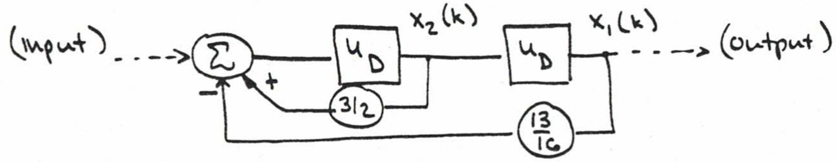
$c' = cT^{-1} = [6 \ \ -6], \ D = 0$



(2.31) A block diagram of the system's internal structure is shown below:

$$P_1 + P_2 = \left(\frac{3}{4} - j\frac{1}{2}\right) + \left(\frac{3}{4} + j\frac{1}{2}\right) = \frac{3}{2}$$

$$P_1 P_2 = \left(\frac{3}{4} - j\frac{1}{2}\right)\left(\frac{3}{4} + j\frac{1}{2}\right) = \frac{9}{16} + \frac{1}{4} = \frac{13}{16}$$



$$\left(r^2 - \frac{3}{2} r + \frac{13}{16} = 0 \quad : \text{char eqn} \right.$$

$$\left. r_{1,2} = \frac{3}{4} \pm j\frac{1}{2} \quad : \text{roots} \right)$$

The state matrix is $A = \begin{bmatrix} 0 & 1 \\ -\frac{13}{16} & \frac{3}{2} \end{bmatrix}$

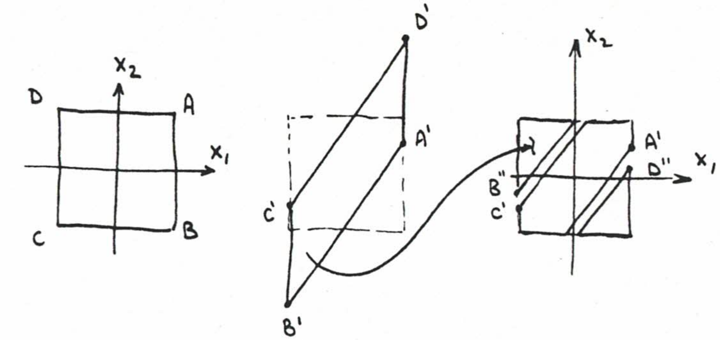
The state space vertices map as

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{11}{16} \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{37}{16} \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ -\frac{5}{16} \end{bmatrix} \quad \text{after overflow}$$

$$A\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{11}{16} \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{37}{16} \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ \frac{5}{16} \end{bmatrix}$$



State map with overflow

As found from simulation of the filter with overflow wrap-around, the behavior may be classified as either

(a) convergence to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) period-2 oscillation of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -.6034 \\ .6034 \end{bmatrix} \rightarrow \begin{bmatrix} .6034 \\ -.6034 \end{bmatrix} \rightarrow \begin{bmatrix} -.6034 \\ .6034 \end{bmatrix} \rightarrow \cdots$$

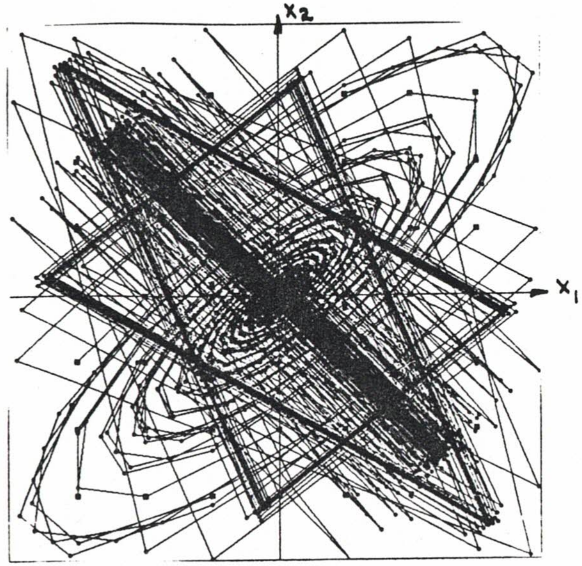
(c) period-3 oscillation of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -.7952 \\ .8556 \end{bmatrix} \rightarrow \begin{bmatrix} .8556 \\ -.0645 \end{bmatrix} \rightarrow \begin{bmatrix} -.0645 \\ -.7952 \end{bmatrix} \rightarrow \begin{bmatrix} -.7952 \\ .8556 \end{bmatrix} \rightarrow \cdots$$

or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .7952 \\ -.8556 \end{bmatrix} \rightarrow \begin{bmatrix} -.8556 \\ .0645 \end{bmatrix} \rightarrow \begin{bmatrix} .0645 \\ .7952 \end{bmatrix} \rightarrow \begin{bmatrix} .7952 \\ -.8556 \end{bmatrix} \rightarrow \cdots$

depending on the initial conditions

A plot of the state trajectories, with various choices of initial conditions, is shown below:



(2.32) By inspection,

$$\begin{aligned} h_0 &= d \\ h_1 &= c_2 \\ h_2 &= c_1 \\ h_3 &= -0.81c_2 \\ h_4 &= -0.81c_1 \\ h_5 &= (-0.81)^2 c_2 \\ h_6 &= (-0.81)^2 c_1 \\ &\dots \\ h_k &= d\delta_k + \begin{cases} c_2(-0.81)^{\frac{k-1}{2}} & k>0,\ \text{odd} \\ c_1(-0.81)^{\frac{k-2}{2}} & k>0,\ \text{even} \end{cases} \end{aligned}$$

Thus

(a) for $h_k = \{1,1,1,\dots\}$

$$c_1 = c_2 = d = 1$$

(b) The steady-state behavior is given by

$$u(k) = 1 \rightarrow y(k) = 1$$

$$u(k) = \cos\frac{k\pi}{2} \rightarrow y(k) = 0$$

ie $H(e^{j0}) = 1$

$H(e^{j\pi/2}) = 0$

From the general form for h_k,

$$H(e^{j\theta}) = \sum_k h_k e^{jk\theta}$$

$$= d + c_2 \sum_{k=1,3,5\dots} (-0.81)^{\frac{k-1}{2}} e^{jk\theta} + c_1 \sum_{k=2,4,6\dots} (-0.81)^{\frac{k-2}{2}} e^{jk\theta}$$

Letting $m = \frac{k-1}{2}$ in the first sum and $\frac{k-2}{2}$ in the second,

$$H(e^{j\theta}) = d + c_2 \sum_0^\infty (-0.81)^m e^{j(2m+1)\theta} + c_1 \sum_0^\infty (-0.81)^m e^{(2m+2)\theta}$$

$$= d + \frac{c_2 e^{j\theta}}{1+0.81e^{j2\theta}} + \frac{c_1 e^{j2\theta}}{1+0.81e^{j2\theta}}$$

ie,

$$H(e^{j\theta}) = \frac{d + c_2 e^{j\theta} + (c_1 + 0.81d)e^{j2\theta}}{1 + 0.81e^{j2\theta}}$$

with

(i) $$H(e^{j0}) = \frac{1.81d + c_2 + c_1}{1 + 0.81} = 1$$

(ii) $$H(e^{j\frac{\pi}{2}}) = \frac{d + jc_2 - (c_1 + 0.81d)}{1 - 0.81} = 0$$

ie (i)' $1.81d + c_2 + c_2 = 1.81$

(ii)' $0.19d + jc_2 - c_1 = 0$

$$\rightarrow \begin{cases} c_2 = 0 \\ c_1 = 0.19d \end{cases}$$

from which

$$d = 0.5$$
$$c_1 = 0.095$$
$$c_2 = 0$$

(2.33) The filter poles are at $\pm j0.9$. For an all-pass filter, we must place the zeros at $\frac{1}{\pm j0.9}$ and adjust the gain:

$$\left| \frac{re^{j\theta} - e^{j\theta_0}}{e^{j\theta} - re^{j\theta_0}} \right| = 1$$

Thus we set

$$H(e^{j\theta}) = \frac{d + c_2 e^{j\theta} + (c_1 + 0.81d)e^{j2\theta}}{1 + 0.81e^{j2\theta}}$$

$$\equiv \frac{0.81 + e^{j2\theta}}{1 + 0.81e^{j2\theta}}$$

from which

$$d = 0.81$$
$$c_1 = 0.19$$
$$c_2 = 0$$

(2.34) From the state-variable equations

$$\underline{x}(n+1) = \underline{\underline{A}}\,\underline{x}(n) + \underline{\underline{B}}\,\underline{u}(n)$$
$$\underline{y}(n) = \underline{\underline{C}}\,\underline{x}(n) + \underline{\underline{D}}\,\underline{u}(n)$$

we first solve for $u(n)$ as

$$u(n) = -\underline{\underline{D}}^{-1}\underline{\underline{C}}\,\underline{x}(n) + \underline{\underline{D}}^{-1}y(n)$$

and substitute above to obtain

$$\underline{x}(n+1) = \underline{\underline{A}}\,\underline{x}(n) - BD^{-1}C\,x(n) + BD^{-1}y(n)$$
$$= [\underline{\underline{A}} - \underline{\underline{B}}\,\underline{\underline{D}}^{-1}\underline{\underline{C}}]\,x(n) + BD^{-1}y(n)$$

Equating to

$$\underline{x}(n+1) = \hat{\underline{\underline{A}}}\,\underline{x}(n) + \hat{\underline{\underline{B}}}\,y(n)$$
$$u(n) = \hat{\underline{\underline{C}}}\,\underline{x}(n) + \hat{\underline{\underline{D}}}\,y(n),$$

We see that the inverse system has state-variable description matrices

$$\hat{\underline{A}} = \underline{A} - \underline{B}\underline{D}^{-1}\underline{C}$$
$$\hat{\underline{B}} = \underline{B}\underline{D}^{-1}$$
$$\hat{\underline{C}} = -\underline{D}^{-1}\underline{C}$$
$$\hat{\underline{D}} = \underline{D}^{-1}$$

with the frequency response

$$G(e^{j\theta}) = \hat{\underline{D}} + \hat{\underline{C}}\,(e^{j\theta}\underline{I} - \hat{\underline{A}})^{-1}\hat{\underline{B}}$$
$$= \underline{D}^{-1} - \underline{D}^{-1}\underline{C}\,(e^{j\theta}\underline{I} - \underline{A} + \underline{B}\underline{D}^{-1}\underline{C})^{-1}\underline{B}\underline{D}^{-1}$$

as given.

$$u(n) \longrightarrow \boxed{H(e^{j\theta})} \xrightarrow{\;y(n)\;} \boxed{G(e^{j\theta})} \longrightarrow u(n)$$

The only constraint is that $\underline{D}^{-1}$ exist: the system may have multiple inputs and outputs (but the same number of each!)

(2.35) Let the inverse filter have impulse response $\{\hat{h}_k\} = \{\hat{h}_0\ \hat{h}_1\ \hat{h}_2 \ldots \hat{h}_6\}$

Then
$$h * \hat{h} = \{1, -1/3\} * \{\hat{h}_0\ \hat{h}_1\ \hat{h}_2 \ldots \hat{h}_6\}$$
$$= \{\hat{h}_0,\ \hat{h}_1 - 1/3\hat{h}_0\ \ \hat{h}_2 - 1/3\hat{h}_1 \ldots \hat{h}_6 - 1/3\hat{h}_5\ \ -1/3\hat{h}_6\}$$

The error energy is given by

$$\epsilon = (1-\hat{h}_0)^2 + (\hat{h}_1 - 1/3\hat{h}_0)^2 + (\hat{h}_2 - 1/3\hat{h}_1) + \ldots + (\hat{h}_6 - 1/3\hat{h}_5) + (-1/3\hat{h}_6)^2$$

Setting $\frac{\partial \epsilon}{\partial h_i} = 0$ for $i = 0, 1, \ldots 6$, we have

$$2(1-\hat{h}_0)(-1) + 2(\hat{h}_1 - 1/3\hat{h}_0)(-1/3) = 0$$
$$2(h_1 - 1/3\hat{h}_0) + 2(\hat{h}_2 - 1/3\hat{h}_1)(-1/3) = 0$$
$$\ldots$$
$$2(\hat{h}_5 - 1/3\hat{h}_4) + 2(\hat{h}_6 - 1/3\hat{h}_5)(-1/3) = 0$$
$$2(\hat{h}_6 - 1/3\hat{h}_5) + 2(-1/3\hat{h}_6) - 1/3 = 0$$

ie

$$(10/9)\hat{h}_0 - (1/3)\hat{h}_1 = 1$$
$$(-1/3)\hat{h}_0 + (10/9)\hat{h}_1 + (-1/3)\hat{h}_2 = 0$$
$$\ldots$$
$$(-1/3)\hat{h}_4 + (10/9)\hat{h}_5 + (-1/3)\hat{h}_6 = 0$$
$$(-1/3)\hat{h}_5 + (10/9)\hat{h}_6 = 0$$

Solving recursively from the bottom up,

$$\hat{h}_5 = (10/3)\hat{h}_6 = (3.333\ldots)\hat{h}_6$$
$$\hat{h}_4 = (10/3)\hat{h}_5 - \hat{h}_6 = (10.111\ldots)\hat{h}_6$$
$$\hat{h}_3 = (10/3)\hat{h}_4 - \hat{h}_5 = (30.370\ldots)\hat{h}_6$$
$$\hat{h}_2 = (10/3)\hat{h}_3 - \hat{h}_4 = (91.123\ldots)\hat{h}_6$$
$$\hat{h}_1 = (10/3)\hat{h}_2 - \hat{h}_3 = (273.374\ldots)\hat{h}_6$$
$$\hat{h}_0 = (10/3)\hat{h}_1 - \hat{h}_2 = (820.125\ldots)\hat{h}_6$$

finally, from the top equation,

$$(10/9)\hat{h}_0 - (1/3)\hat{h}_1 = 820.125\,\hat{h}_6 = 1$$

$$\Rightarrow \hat{h}_6 = 0.001219$$

Thus

$$\{\hat{h}_k\} = \{1, 0.3333, 0.1111, 0.0370, 0.0123, 0.0041, 0.0012\}$$

A more elegant solution would proceed as

$$\begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & \beta & \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & \beta & \alpha & \beta & 0 \\ 0 & 0 & 0 & 0 & \beta & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & \beta & \alpha \end{bmatrix} \begin{bmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \\ \hat{h}_4 \\ \hat{h}_5 \\ \hat{h}_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \alpha &= 1+(1/3)^2 \\ \beta &= -1/3 \end{aligned}$$

[cf (2.8.13)]

hence $$\underline{\hat{h}} = [\quad]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= 1^{st} \text{ column of } \begin{bmatrix} \alpha & \beta & 0 & \cdots \\ \beta & \alpha & \beta & \cdots \\ \cdots & & \beta & \alpha\ \beta \\ 0 & 0 & 0 \cdots 0 & \beta\ \alpha \end{bmatrix}^{-1}$$

Chapter 3

3.1 (a)
$$r^2+3r+2=0$$
$$(r+1)(r+2)=0 \rightarrow r_1=-1,\ r_2=-2$$
$$y(t) = c_1e^{-t} + c_2e^{-2t}$$
$$\left.\begin{aligned} y(0) &= c_1+c_2 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1-2c_2 = 0 \end{aligned}\right\} \begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$
$$y(t) = 2e^{-t} - e^{-2t}$$

(b)
$$r^2+2r+1=0$$
$$(r+1)^2=0 \rightarrow r_1=r_2=-1$$
$$y(t) = c_1e^{-t} + c_2te^{-t}$$
$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} &= -c_1+c_2 = 0 \end{aligned}\right\} \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$
$$y(t) = (1+t)e^{-t}$$

(c)
$$r^3+4r^2+5r+2=0$$
$$(r+2)(r+1)^2=0 \rightarrow r_1=-2,\ r_2=r_3=-1$$
$$y(t) = c_1e^{-2t} + c_2e^{-t} + c_3te^{-t}$$