Chapter 1

1.1 (a) $T(u(t)) = \begin{cases} 0, & t < T_0 \\ \alpha u(t), & t \ge T_0 \end{cases}$

Thus $T\{a u_1(t) + b u_2(t)\} = \begin{cases} 0, & t < T_0 \\ \alpha(a u_1(t) + b u_2(t)), & t \ge T_0 \end{cases}$

$= a\,T(u_1(t)) + b\,T(u_2(t))$. $\therefore$ Linear

(b) System is not linear. Counter example:

$u(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $w(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $a = b = 1$.

Then $\min\{1+2, 3+1\} = 3$, but $\min\{1,3\} + \min\{2,1\} = 2$

(c) $T\{u(t)\} = \begin{cases} \alpha u(t) + u(t^2), & t \ge 0 \\ 0, & t < 0 \end{cases}$

And so, $T\{a u_1(t) + b u_2(t)\} = \begin{cases} \alpha(a u_1(t) + b u_2(t)) + a u_1(t^2) + b u_2(t^2), & t \ge 0 \\ 0, & t < 0 \end{cases}$

$= a\,T(u_1(t)) + b\,T(u_2(t))$ $\therefore$ Linear

(d) $y(k) = T(u(k)) = \sum_{n=k}^{k+2} k^2 u(n)$

And so, $T(a u_1(k) + b u_2(k)) = \sum_{n=k}^{k+2} k^2 (a u_1(n) + b u_2(n))$

$= a \sum_{n=k}^{k+2} u_1(n) k^2 + b \sum_{n=k}^{k+2} k^2 u_2(n)$

$= a\,T(u_1(k)) + b\,T(u_2(k))$ $\therefore$ Linear

(e) $y(k) = T(u(k)) = \alpha u(k) + \beta u(k-1) + \gamma[u(k-2)]^2$

And so, $T(a u_1(k) + b u_2(k)) = \alpha(a u_1(k) + b u_2(k)) + \beta(a u_1(k-1) + b u_2(k-1)) + \gamma(a u_1(k-2) + b u_2(k-2))^2$

$\ne a\,T(u_1(k)) + b\,T(u_2(k))$

Because of the cross terms in the squared term $\therefore$ not Linear

1.2 $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ -1 & t & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$ $\quad \underline{y}(t) = \underline{A}(t)\,\underline{u}(t)$

(a) System is linear since: $T(a\underline{u}(t) + b\underline{w}(t)) = \underline{A}(t)(a\underline{u}(t) + b\underline{w}(t))$

$= a\,\underline{A}(t)\underline{u}(t) + b\,\underline{A}(t)\underline{w}(t) = a\,T(\underline{u}(t)) + b\,T(\underline{w}(t))$.

(b) System is not time invariant because a parameter defining the transformation varies with time. ($\underline{A}(t)$ is a function of t.).

1.3 $T(u(t)) = u(t) \cdot f\left(\frac{u(t)}{u(0)}\right)$ where $f(\cdot)$ is a non linear function. For example:

$T(u(t)) = u(t)\left(\frac{u(t)}{u(0)}\right)^2$

$T(u(t)) = u(t)\,\exp\left(\frac{u(t)}{u(0)}\right)$

These two transformations don't satisfy superposition, & hence are non linear.

1.4

(a) $y(t) = T(a(t)) = a(t)\,p(t) = \begin{cases} 0, & t \in T_{open} \\ a(t), & t \in T_{closed} \end{cases}$

$T(\alpha a_1(t) + \beta a_2(t)) = \begin{cases} \alpha a_1(t) + \beta a_2(t), & t \in T_{closed} \\ 0, & t \in T_{open} \end{cases}$

$= \alpha\,T(a_1(t)) + \beta\,T(a_2(t))$ $\therefore$ Circuit is linear.

(b) system is clearly time varying.

Note: This is a time varying system which modulates the incoming waveform, $a(t)$. Contrary to what is often stated, modulation can be accomplished with linear circuits. It is true that no linear, time-invariant systems can produce modulation.

1.5 System 1 : $T(u(t)) = \frac{d}{dt}u(t)$; $T(a u_1(t) + b u_2(t)) = \frac{d}{dt}(a u_1(t) + b u_2(t))$

(a) $= a\frac{d}{dt}u_1(t) + b\frac{d}{dt}u_2(t)$

$= aT(u_1(t)) + bT(u_2(t))$ $\therefore$ system 1 is linear.

System 2 : $T(u(t)) = t\,u(t)$

$T(a u_1(t) + b u_2(t)) = t\,(a u_1(t) + b u_2(t)) = a t u_1(t) + b t u_2(t)$

$= a\,T(u_1(t)) + b\,T(u_2(t))$ $\therefore$ System 2 is linear

(b) $u(t) \rightarrow \boxed{1} \rightarrow \boxed{2} \rightarrow$ $y(t) = t.\frac{d}{dt}u(t)$

$u(t) \rightarrow \boxed{2} \rightarrow \boxed{1} \rightarrow$ $y(t) = \frac{d}{dt}[t\,u(t)] = u(t) + t\frac{d}{dt}u(t)$

$\therefore$ Systems do not commute.

1.6 Try a few simple cases.

 $n=1$, $y_1 = 2$

 $n=2$, $y_2 = 4$

 $n=3$, $y_3 = 7$

 $n=4$, $y_4 = 11$

As each additional line is added, n regions are added to the previous number of regions. And so we have

$$y_n = y_{n-1} + n \quad \text{or} \quad y_n - y_{n-1} = n \quad \text{with } y_0 = 1.$$

1.7 S_k = sum on deposit after year k

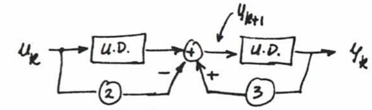
S_0 = initial deposit

Then $S_{k+1} = S_k + rS_0$, $k = 0, 1, 2, \ldots$

Thus the first few terms are : S_0, $S_1 = S_0 + rS_0 = S_0(1+r)$,

$S_2 = S_1 + rS_0$
$= S_0(1+r) + rS_0$
$= S_0(1+2r)$
, $S_3 = S_2 + rS_0$
$= S_0(1+3r)$
, ... , $\underline{\underline{S_k = S_0(1+kr_0)}}$, ...

1.8



At the output of the summer : $y_{k+1} = -2u_k + u_{k-1} + 3y_k$

1.9

(a) $y_k = \frac{u_k + u_{k-1} + u_{k-2} + u_{k-3}}{4}$; u_{k-i}, $i > 0$ are "previous values" of the input sequence.

(b) $y_k = u_k + 2u_{k-1} + 2u_{k-2} + u_{k-3}$

1.10

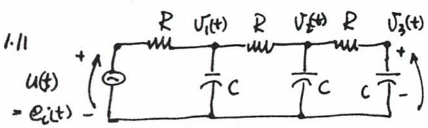
$$y_k = \frac{u_k + u_{k-1} + u_{k+1} + y_{k-1}}{4}$$

u_{k-1} : previous value of data

u_{k+1} : future or next value of data

y_{k-1} : (one) previous output value

1.11



Writing node eqs. we have

$\frac{v_1(t) - u(t)}{R} + \frac{v_1(t) - v_2(t)}{R} + C\frac{dv_1(t)}{dt} = 0$; $\frac{dv_1(t)}{dt} = -\frac{2}{RC}v_1(t) + \frac{1}{RC}v_2(t) + \frac{u(t)}{RC}$

$\frac{v_2(t) - v_1(t)}{R} + \frac{v_2(t) - v_3(t)}{R} + C\frac{dv_2(t)}{dt} = 0$; $\frac{dv_2(t)}{dt} = \frac{1}{RC}v_1(t) - \frac{2}{RC}v_2(t) + \frac{1}{RC}v_3(t)$

$\frac{v_3(t) - v_2(t)}{R} + C\frac{dv_3(t)}{dt} = 0$; $\frac{dv_3(t)}{dt} = \frac{1}{RC}v_2(t) - \frac{1}{RC}v_3(t)$

In vector form

$$\begin{bmatrix} \frac{dv_1(t)}{dt} \\ \frac{dv_2(t)}{dt} \\ \frac{dv_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -2/RC & 1/RC & 0 \\ 1/RC & -2/RC & 1/RC \\ 0 & 1/RC & -1/RC \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 1/RC \\ 0 \\ 0 \end{bmatrix} u(t) \quad ; \; \underline{v}'(t) = A\,\underline{v}(t) + B\,u(t)$$

with $y(t) = v_3(t)$.

To obtain a differential equation relating $u(t)$ and $y(t)$ we can proceed as follows: We have the following three equations

$$\left.\begin{aligned} v_1'(t) &= -\frac{2}{RC} v_1(t) + \frac{1}{RC} v_2(t) + \frac{u(t)}{RC} \\ v_2'(t) &= \frac{1}{RC} v_1(t) - \frac{2}{RC} v_2(t) + \frac{1}{RC} y(t) \\ y'(t) &= \frac{1}{RC} v_2(t) - 1/RC\, y(t) \end{aligned}\right\} \text{ Since } y(t) = v_3(t).$$

To obtain a differential equation relating $u(t)$ and $y(t)$ only we, of course, must eliminate the variables $v_1(t)$, $v_2(t)$ and their derivatives. Perhaps the easiest method is to use operator notation or Laplace transform notation to express the equations in the form

$$s V_1(s) = -\frac{2}{RC} V_1(s) + \frac{1}{RC} V_2(s) + \frac{U(s)}{RC}$$

$$s V_2(s) = \frac{1}{RC} V_1(s) - \frac{2}{RC} V_2(s) + \frac{1}{RC} Y(s)$$

$$s Y(s) = \frac{1}{RC} V_2(s) - \frac{1}{RC} Y(s)$$

Now eliminate $V_1(s)$ and $V_2(s)$ to obtain an equation

$$s^3 Y(s)(RC)^2 + s^2 Y(s) RC - sY(s) - 2\frac{Y(s)}{RC} = \frac{U(s)}{RC}$$

Which results in the differential equation:

$$(RC)^2 \frac{d^3 y(t)}{dt^3} + (RC)\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - \frac{2}{RC} y(t) = \frac{u(t)}{RC}$$

Notice that in contrast to discrete-time systems this process for continuous-time systems is nontrivial. For discrete-time systems the "equations of motion" are usually very easy to obtain.

1.12 Following the style of Figure 1.4.1 (b)

```
   READ X1K, X2K
10 READ UK
   YK = 0.5 * X1K - 1.5 * X2K
   PRINT YK
   XT = UK + 0.375 * X1K - 0.25 * X2K
   X1K = X2K
   X2K = XT
   GO TO 10
```

1.13 From the block diagram,

(a) $y(k) = \frac{1}{2} x_1(k) - \frac{3}{2} x_1(k+1)$

(b) $x_1(k+2) = u(k) - \frac{1}{4} x_1(k+1) + \frac{3}{8} x_1(k)$

ie $x_1(k+2) + \frac{1}{4} x_1(k+1) - \frac{3}{8} x_1(k) = u(k)$

By substitution,

(a') $y(k+2) + \frac{1}{4} y(k+1) - \frac{3}{8} y(k)$

$$= \frac{1}{2} x_1(k+2) - \frac{3}{2} x_1(k+3) + \frac{1}{8} x_1(k+1) - \frac{3}{8} x_1(k+2) - \frac{3}{16} x_1(k) + \frac{9}{16} x_1(k+1)$$

(b') $\frac{1}{2}u(k) - \frac{3}{2}u(k+1)$

$$= \frac{1}{2}x_1(k+2) + \frac{1}{8}x_1(k+1) - \frac{3}{16}x_1(k)$$
$$- \frac{3}{2}x_1(k+3) - \frac{3}{8}x_1(k+2) + \frac{9}{16}x_1(k+1)$$

Since the RHS of (a') and (b') are identical,

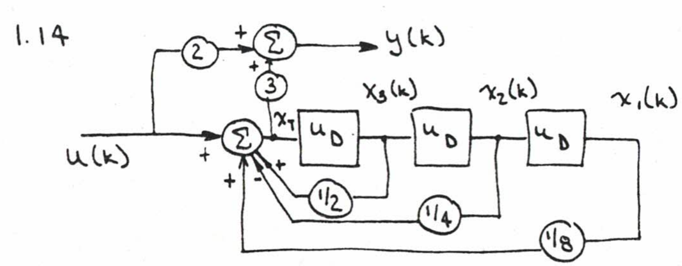
$$y(k+2) + \frac{1}{4}y(k+1) - \frac{3}{8}y(k) = \frac{1}{2}u(k) - \frac{3}{2}u(k+1)$$

Note the equivalent operator notation (cf. Chapter 2):

$$y(k) = \left(\frac{1}{2} - \frac{3}{2}S\right)x_1(k)$$
$$u(k) = \left(S^2 + \frac{1}{4}S - \frac{3}{8}\right)x_1(k)$$

$$\Rightarrow \left(S^2 + \frac{1}{4}S - \frac{3}{8}\right)y_k = \left(\frac{1}{2} - \frac{3}{2}S\right)u(k)$$

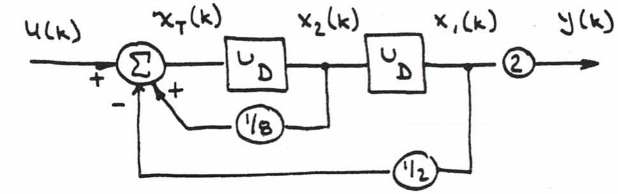
1.14



1.15

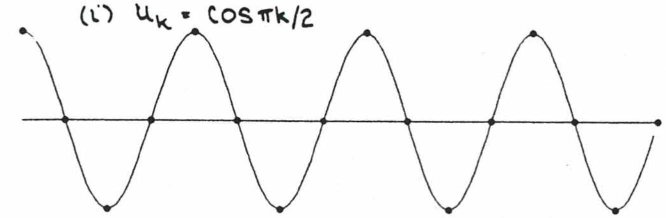
```
    READ XK1, XK2
10  READ U
    Y = 2 * XK1
    XT = U - 0.5 * XK1 + 0.125 * XK2
    XK1 = XK2
    XK2 = XT
    GO TO 10
```

1.16

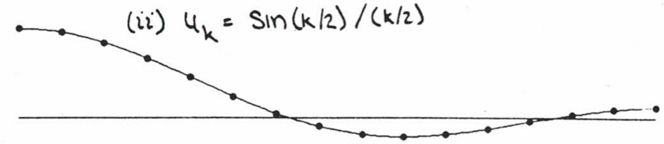


1.17 (a) T = 1/2

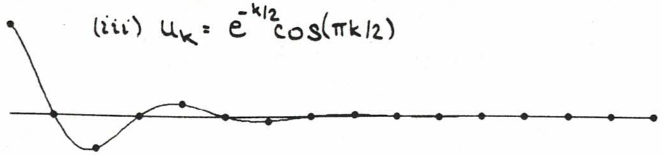
(i) $u_k = \cos \pi k/2$



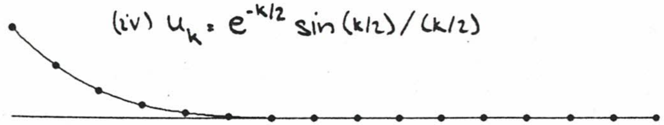
(ii) $u_k = \sin(k/2)/(k/2)$



(iii) $u_k = e^{-k/2}\cos(\pi k/2)$

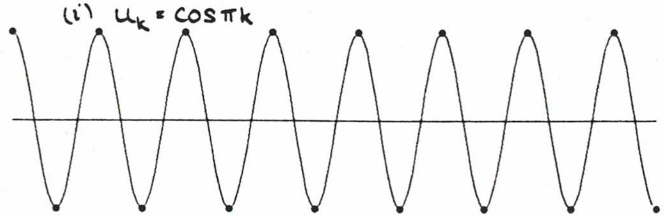


(iv) $u_k = e^{-k/2}\sin(k/2)/(k/2)$

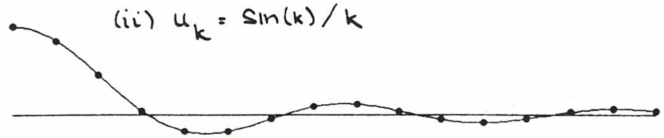


(b) $T = 1$

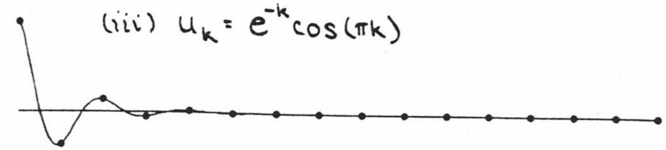
(i) $u_k = \cos \pi k$



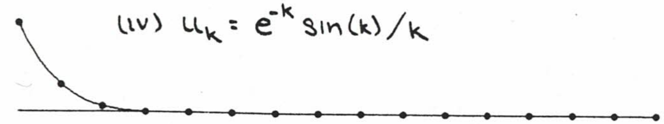
(ii) $u_k = \sin(k)/k$



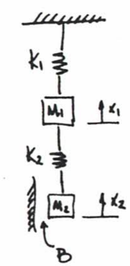
(iii) $u_k = e^{-k}\cos(\pi k)$



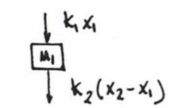
(iv) $u_k = e^{-k}\sin(k)/k$



1.18

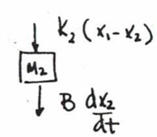


Free body diagrams:



Using $F = ma$ we have:

$$-k_1 x_1 - k_2(x_1 - x_2) = M_1 \frac{d^2 x_1}{dt^2} \quad (1)$$



$$k_2(x_1 - x_2) - B\frac{dx_2}{dt} = M_2 \frac{d^2 x_2}{dt^2} \quad (2)$$

Eqs (1) and (2) are the equations of motion for the mechanical system shown.